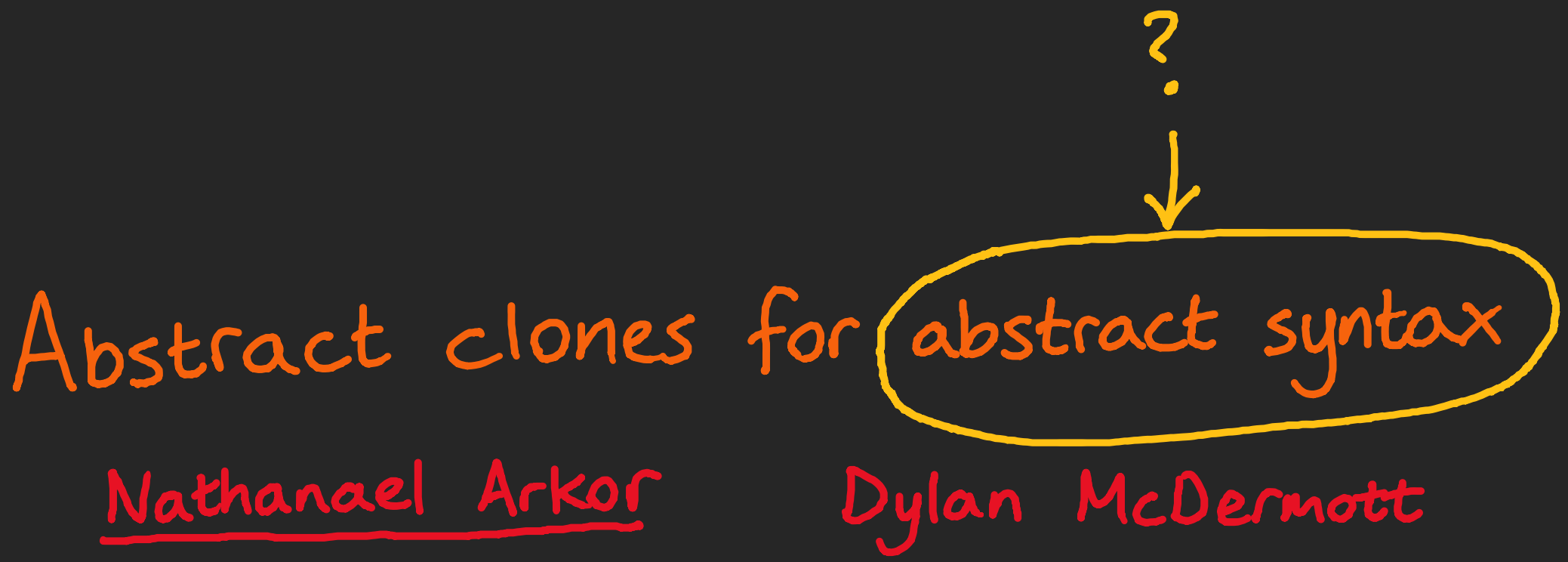
Abstract clones for abstract syntax

Nathanael Arkor Dylan McDermott

(FSCD 2021)



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equational theory (i.e. universal algebra)

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multisorted equational theory (i.e. multisorted universal algebra)

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multisorted equational theory + variable binding

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There exist formalisms in the literature to capture this structure.

- Σ-monoids of Fiore-Plotkin-Turi '99.
- Second-order algebraic theories of Fiore-Mahmoud '10.
- Structured monads and relative monads, cf. Ahrens, Hirschowitz, Hirschowitz, Lafont, Maggesi.
- Structured cartesian multicategories, à la Arkor-Fiore '20.

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(*) Highly categorical

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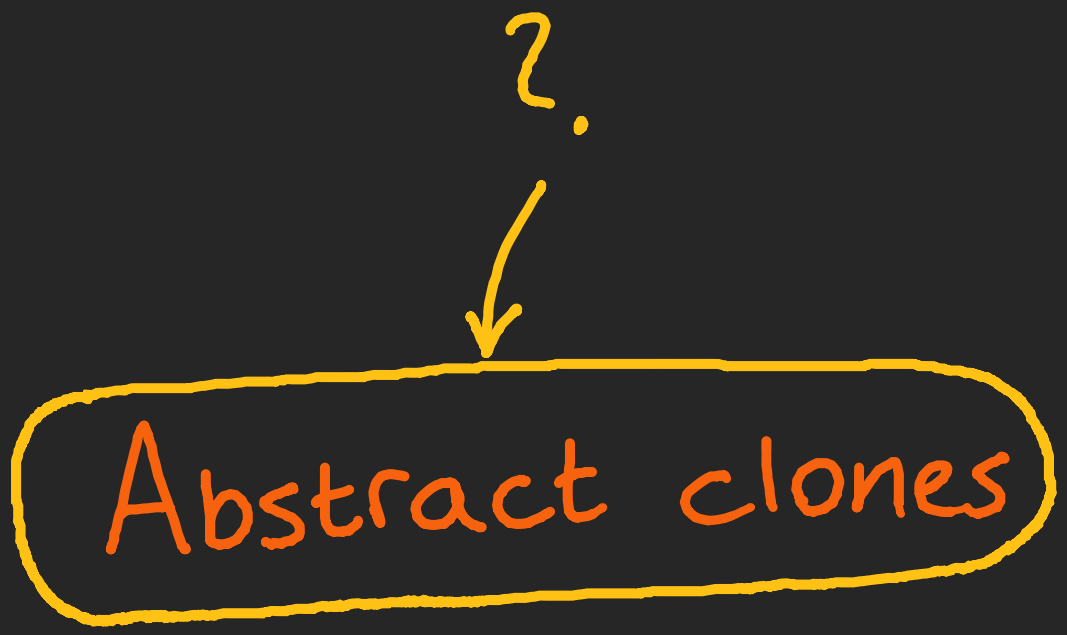
Yes... and in fact this approach has appeared in limited cases in the literature previously (Mahmoud '11, Hyland '17). However, no unified account exists. This is the purpose of our paper.



for abstract syntax

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We can equivalently consider a term

$$x_1, \ldots, x_n \vdash t$$

as an operation

$$t : X^n \longrightarrow X$$

and in this way abstract clones capture algebra.

Abstract clones are equivalent to many other concepts (from category theory), for instance:

- Algebraic theories
- Cartesian operads
- Finitary monads on Set
- $(\mathrm{FinSet} \hookrightarrow \mathrm{Set})$-relative monads
- Substitution algebras of Fiore-Plotkin-Turi '99
- Monoids for the substitution tensor product

which are all equivalent to the traditional concept of equational theory.

Abstract clones

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  - a family of elements for each context representing variable projections
    $$x_i \in X(x_1 : A_1 , \ldots , x_n : A_n \; ; \; A_i) \quad (i \leq n)$$
  - a function representing simultaneous substitution
    $$\frac{x_1 : A_1 , \ldots , x_n : A_n \vdash t : B \qquad \Gamma \vdash u_1 : A_1 \quad \cdots \quad \Gamma \vdash u_n : A_n}{\Gamma \vdash t \, [x_1 \mapsto u_1 , \ldots , x_n \mapsto u_n]}$$

satisfying laws axiomatizing substitution.

How do abstract clones capture algebraic structure? Consider the following two presentations of a binary operator:

$$\frac{}{x, y \vdash f(x, y)} \qquad \frac{\Gamma \vdash s \qquad \Gamma \vdash t}{\Gamma \vdash f(s, t)}$$

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The terms of the unityped λ-calculus form an abstract clone Λ.

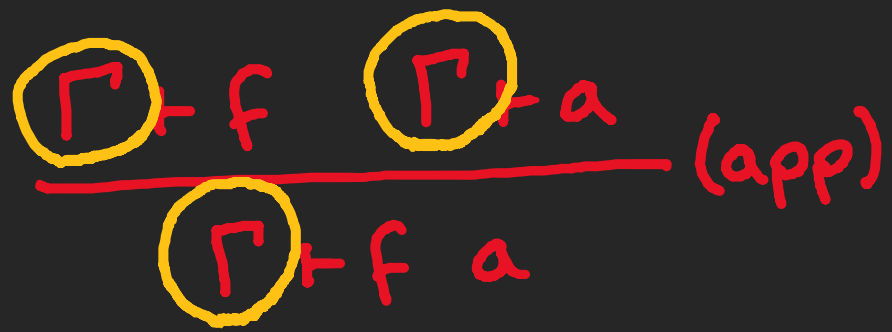
$$\frac{\Gamma \vdash f \quad \Gamma \vdash a}{\Gamma \vdash f\ a}\,(app)$$

Application is an algebraic operation and is therefore captured by the clone structure, as an element

$$app \in \Lambda(*, * \,;\, *)$$

which induces a family of functions

$$[\![app]\!] : \Lambda(\Gamma\,;\, *) \times \Lambda(\Gamma\,;\, *) \longrightarrow \Lambda(\Gamma\,;\, *) \qquad (\forall \Gamma)$$



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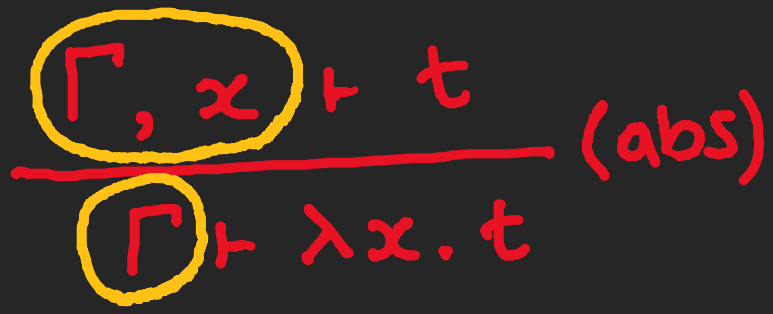
However, abstraction is not an algebraic operation, since it involves a variable binding. Therefore, it is not captured by the clone structure.



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Observe that, if the abstraction was captured by the abstract clone, it would induce a family of functions

$$[\![\text{abs}]\!] : \Lambda(\Gamma, * \,;\, *) \longrightarrow \Lambda(\Gamma \,;\, *) \qquad (\forall \Gamma)$$

In fact, it suffices to axiomatize this derived structure, to capture variable binding.

Idea

Universal algebraic structure is captured by algebraic structure on sets:

$$M \times M \xrightarrow{\otimes} M$$

(+ equations)

Variable-binding structure is captured by algebraic structure on abstract clones:

$$\Lambda(\Gamma, * \,;\, *) \xrightarrow{\text{abs}} \Lambda(\Gamma \,;\, *) \qquad (\forall \Gamma)$$

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need not be homogeneous

Therefore, to capture simple type theories, we may consider abstract clones equipped with algebraic structure. This leads to a conceptually elegant and practical approach to abstract syntax that avoids the categorically sophisticated constructions arising in other approaches.

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What does this look like, practically?

A simple type theory is usually presented by means of natural deduction rules (formation rules, introduction rules, elimination rules, computation rules, etc.). Formally, this corresponds to a second-order presentation:

$$(\Sigma, E)$$

Σ: signature of variable-binding operators

E: equations on derived terms

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The algebras for a presentation are abstract clones equipped with algebraic structure corresponding to the rules described by the presentation.

Algebras for a second-order presentation are the simple type theories interpreting the rules of the presentation. We are usually interested in a canonical algebra: the one generated freely from the rules. This is the free algebra for the presentation, and may be understood as the abstract syntax for our programming language.

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Every presentation admits a free algebra, given by an inductive construction using the second-order equational logic of Fiore and Hur.

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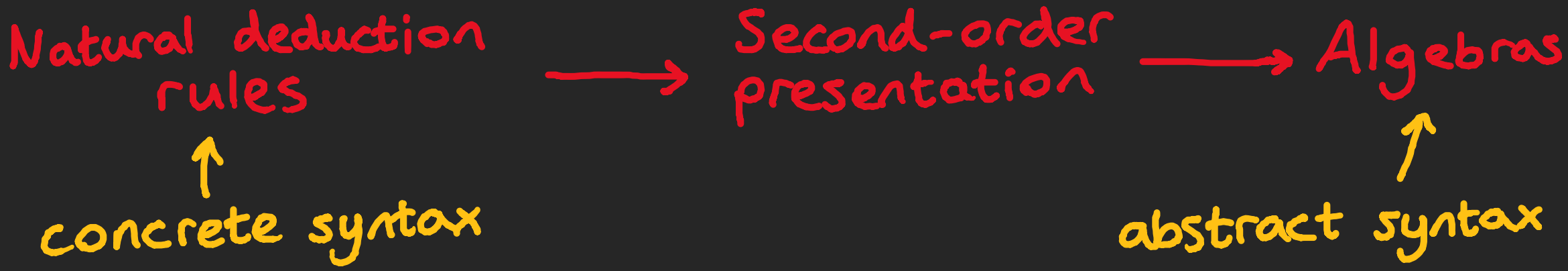
We show how abstract clones with algebraic structure can be used to perform logical relations arguments. In particular, we prove

- adequacy for the set-theoretic model of the STLC
- normalization for the STLC
- ... with global state

High-level overview



We may then prove theorems about the programming language using more abstract and elegant techniques, facilitated by general metatheorems, such as our induction principle for second-order syntax.

Thesis

Abstract clones provide a viable and practical approach to abstract syntax especially well-suited for applications in which category-theoretic techniques are avoided.

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For the complete framework, more details, and applications, see the paper:

Abstract clones for abstract syntax

Thanks for listening!