Higher-order algebraic theories

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Algebraic theories give a presentation-free categorical formulation of universal algebraic structure: objects equipped with first-order operators, subject to equational laws. Similarly, higher-order algebraic theories describe objects equipped with higher-order, variable-binding operators, such as logical quantifiers or λ-abstraction. While higher-order structures abound in mathematics and computer science, there exists no systematic treatment in the spirit of that for first-order structure. This has led to a proliferation of variations of higher-order theory, and consequently a lacklustre general understanding. We take the first steps to rectify this, defining a notion of multisorted higher-order algebraic theory and carrying out a development analogous to that of the first-order setting. In addition to unifying various previous notions, we (1) establish a correspondence between higher-order algebraic theories and a class of (relative) monads, whose algebras describe the closed-term structure of the corresponding theories; (2) prove that the categories of higher-order algebraic theories, and of the term algebras for a higher-order algebraic theory, are locally strongly finitely presentable; (3) give a new explanation for the apparent asymmetry between models of algebraic theories in the category of sets, and models in arbitrary cartesian categories.

1 INTRODUCTION

The notion of algebraic theory (or Lawvere theory) was introduced in Lawvere's [1963] seminal thesis as a categorical, presentation-free axiomatisation of universal algebraic structure. Though examples of universal algebraic structure abound, there are many structures throughout mathematics that cannot be described thus and, since then, many extensions or variations of algebraic theories have arisen [Bénabou 1968; Freyd 1972; Power 1999]. One such variation is that of *second-order algebraic theory* [Fiore and Mahmoud 2010], which extends the structure of algebraic theories with a notion of variable-binding operator. This is a rich setting, covering many examples of simple type theories, such as the unityped and simply-typed λ-calculi [Church 1940], and the computational λ-calculus [Moggi 1989]; as well as structures such as predicate logic, and partial differentiation [Plotkin 2020]. Second-order algebraic theories are more conservative than cartesian-closed categories, which may also be used to describe structures with variable-binding operators, requiring only a set of exponentiable objects, and may thus be modelled even in categories that are not cartesian-closed. However, second-order algebraic theories are poorly understood, having undergone little general development. While Fiore and Mahmoud [2010] introduce second-order algebraic theories, establishing their equivalence to the monosorted second-order presentations of Fiore and Hur [2010], and prove them to be conservative over first-order algebraic theories, there are many questions left unanswered [Fiore and Mahmoud 2010, Section 7]. In particular, we should like to know to what extent the classical results in the first-order setting carry through (cf. Adámek, Rosický, and Vitale [2010]).

This work may be seen as a starting point for a systematic treatment of second-order algebraic theories in the spirit of that for first-order algebra. However, to fully appreciate where second-order algebraic theories stand in relation to similarly motivated concepts, such as typed λ-calculi [Lambek 1980; Lambek and Scott 1988], higher-order universal algebra [Meinke 1992, 1995; Poigné 1986], higher-order abstract syntax [Pfenning and Elliott 1988], and parameterised algebraic theories [Staton 2013a], we consider multisorted *higher-order algebraic theories*, subsuming both n^{th}-order algebraic theories, for $n \in \mathbb{N}$, and ω-order algebraic theories, whose operations have unrestricted

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order. In this generality, we carry out a development of multisorted higher-order algebraic theories, their equational logic, presentations, and models. Much of the structure known to be present in the first-order setting is shown also to exist in the higher-order setting: for example, we show that the category of n^{th}-order algebraic theories is locally strongly presentable and hence complete and cocomplete. In addition, we establish a correspondence between $(n+1)^{\text{th}}$-order algebraic theories and a class of monads on the category of n^{th}-order algebraic theories, which specialises, when $n = 1$, to the classical correspondence between algebraic theories and finitary monads on **Set**.

Throughout, we have endeavoured to give intuition for the constructions and correspondences that appear in the development; though the study of algebraic theories is well-established, it can be difficult to find philosophical justification for certain phenomena. For instance, we discuss the conceptual distinction between models of first-order algebraic in **Set**, and in arbitrary cartesian categories; and give an explicit description of the monad induced by a higher-order algebraic theory that is new even in the first-order setting.

1.1 Overview and contributions

We begin in Section 2 by discussing the ways in which higher-order algebraic theories may be seen as extensions of first-order equational logic: this frames our setting in a wider context, relating it to various similarly-motivated concepts. In Section 3, we introduce our preferred equational logic for higher-order algebraic theories, along with their presentations, and give various examples. We then move towards the categorical development of higher-order algebraic theories in the style of Lawvere. Section 4 details a non-syntactic construction of free cartesian-closed categories, which is a prerequisite to define higher-order algebraic theories. In Section 5, we introduce higher-order algebraic theories, prove them to be equivalent to higher-order presentations, and describe their models and term algebras. In particular, we explain an asymmetry between models for algebraic theories and algebras for finitary monads on **Set**. Next, we establish a monad–theory correspondence for higher-order algebraic theories. We show in Section 6 that higher-order algebraic theories correspond to a class of relative monads. In Section 7, the categories of higher-order algebraic theories are shown to be locally strongly presentable, which allows us to obtain an equivalence with a class of monads in Section 8. We briefly discuss 0^{th}-order algebraic theories in Section 9. In in **??**, we define higher-order abstract clones, standing in relation to higher-order algebraic theories as abstract clones do to algebraic theories. Finally, **??**, we examine the categories of multisorted higher-order algebraic theories, combining S-sorted higher-order algebraic theories for fixed sets of sorts S, and relate them to arbitrary cartesian-closed categories.

1.2 Conventions

We will use the term *higher-order algebraic theory* to refer to n^{th}-order algebraic theories, for arbitrary $n \in \mathbb{N}_\omega$, where $\mathbb{N}_\omega$ is the total order of extended natural numbers $(\mathbb{N} + \{\omega\}, \leq)$. We fix a set S of sorts B throughout. The category $\mathbf{Law}_0(S)$ of S-sorted 0^{th}-order algebraic theories is defined to be the category $\mathbf{Set}^S$ of S-indexed sets; this definition is formally justified in Section 9. We use $\varkappa$ to denote coprojections.

2 PERSPECTIVES & RELATED WORK

Before beginning the technical development, we outline several perspectives from which higher-order algebraic theories may be viewed: each of these viewpoints has appeared separately in the literature, though we know of no source in which the connections are explicated. Indeed, many of the developments from each perspective appear to exist in isolation from the others, and for this reason it is difficult for a non-expert to build a holistic picture of the field. We suggest that it is by considering the perspective of n^{th}-order algebraic theories, as a bridge between first-order algebraic theories and ω-order algebraic theories, that the full picture is made most clear.

2.1 Higher-order natural deduction

Universal algebra, or more precisely its associated first-order equational logic, may be seen as a basic natural deduction system in which we have two judgements, for the well-formedness of terms, and for their equality. The (term) operators of an algebra take a sequence of terms, the *operands*, and form a new term. One may present an operator syntactically by an inference rule of the form:

$$\frac{\vdash t_1 \qquad \cdots \qquad \vdash t_n}{\vdash \mathsf{f}(t_1, \ldots, t_n)} \tag{1}$$

We read this inference rule as "if there exist well-formed terms t_1 through to t_n, then we may form a new well-formed term $\mathsf{f}(t_1, \ldots, t_n)$"; we think of f as being an operator that we apply to the operands t_1 through to t_n. One may consider variations on first-order equational logic by modifying the structure of these inference rules. For example, associating a sort (or type) to each term leads to the notion of multisorted algebraic theory [Bénabou 1968]. Here, inference rules may only be applied if the operands have the correct type. For instance, the action of a monoid may be presented by an inference rule of the form:

$$\frac{\vdash m : \mathsf{M} \qquad \vdash x : \mathsf{X}}{\vdash \mathsf{act}(m, x) : \mathsf{X}} \tag{2}$$

We read this inference rule as "if there exists a well-formed term m of type M and a well-formed term x of type X, then we may form a new well-formed term $\mathsf{act}(m, x)$ of type X". In fact, in the absence of ill-formed terms (which arise only when one considers concrete syntax, formed through string concatenation from basic symbols), we may drop "well-formed" and simply talk about unqualified "terms".

Higher-order equational logic arises when one considers operators that may themselves take operators, rather than terms, as their operands. A second-order operator may therefore be presented by an inference rule whose premisses are themselves (first-order) inference rules. For instance, consider the following second-order inference rule:

$$\frac{\left(\dfrac{\vdash t_{1\,1} \qquad \cdots \qquad \vdash t_{1\,n_1}}{\vdash f_1(t_{1\,1}, \ldots, t_{1\,n_1})}\right) \qquad \cdots \qquad \left(\dfrac{\vdash t_{n\,1} \qquad \cdots \qquad \vdash t_{n\,n_n}}{\vdash f_n(t_{n\,1}, \ldots, t_{n\,n_n})}\right)}{\vdash \mathsf{g}(f_1, \ldots, f_n)} \tag{3}$$

We read this inference rule as "if there exist inference rules that takes $t_{i\,1}$ through to $t_{i\,n_i}$ and forms a term $f_i(t_{i\,1}, \ldots, t_{i\,n_i})$, for $1 \leq i \leq n$, then we may form a term $\mathsf{g}(f_1, \ldots, f_n)$"; we think of g as an operator that we apply to the inference rules f_1 through to f_n. Note that when we say "inference rule" for f_i, we really mean any possible derivation of a term given terms $t_{i\,1}$ through to $t_{i\,n_i}$: we permit the composition of inference rules by grafting conclusions of one inference rule to a premiss of another, to form open derivations of terms.

Similarly, we may consider third-order operators, which take second-order operators as their operands, and so on for arbitrary $n \in \mathbb{N}$. Note that nullary $(n+1)^{\text{th}}$-order operators (that is, operators that take no operands) are equivalently n^{th}-order operators: in this way, $(n+1)^{\text{th}}$-order operators strictly subsume n^{th}-order operators. From this perspective, 0^{th}-order operators are equivalently constants. We may define an ω-order operator to be an n^{th}-order operator for any $n \in \mathbb{N}$.

These higher-order operators may be motivated for the purpose of metatheoretic reasoning: by ascending to a higher order, it is possible to perform operations on operations (of a lower order). For example, we can describe a second-order operator that formally adds an inverse for a unary first-order operator:

$$\frac{\left(\dfrac{\vdash x}{\vdash f(x)}\right) \qquad \vdash t}{\vdash \mathsf{inv}(f, t)} \qquad \frac{\left(\dfrac{\vdash x}{\vdash f(x)}\right) \qquad \vdash t}{\vdash \mathsf{inv}(f, f(t)) \equiv t} \qquad \frac{\left(\dfrac{\vdash x}{\vdash f(x)}\right) \qquad \vdash t}{\vdash f(\mathsf{inv}(f, t)) \equiv t} \tag{4}$$

In practice, as evidenced by even the simple second-order operator above, higher-order equational logic presented recursively in this style quickly becomes unwieldy, but it nevertheless gives a useful intuition. This approach was explored from a syntactic perspective by Schroeder-Heister [1984].

2.2 Equational logics with metavariables

First-order operators are usually defined as symbols that take terms as operands: for any compatible choice of operand terms, we may form a new term, which is thought of as the application of that operator. However, there is another choice: we may instead define operators as symbols parameterised by a context of variables, such as in the following inference rule:

$$\frac{}{x_1, \ldots, x_n \vdash \mathsf{f}} \tag{5}$$

We read this inference rule as "we may form a term f in any context with n variables". We can understand f as some term containing free variables (this is called an *open term*): to apply the operator f, we substitute each of the variables x_1 through to x_n by terms t_1 through to t_n, as in the following:

$$\frac{\vdash t_1 \qquad \cdots \qquad \vdash t_n}{\vdash \mathsf{f}[{}^{t_1}/_{x_1}] \cdots [{}^{t_n}/_{x_n}]} \tag{6}$$

Note that this inference rule has the same form as (1): the difference is simply in whether we form a compound term $\mathsf{f}(t_1, \ldots, t_n)$ or substitute for free variables in an open term $\mathsf{f}[{}^{t_1}/_{x_1}] \cdots [{}^{t_n}/_{x_n}]$. In an appropriate, formal sense, these perspectives are equivalent; one may consider premisses $\vdash t_i$ in empty contexts to correspond to variables x_i, and vice versa. It is natural to then ask whether there is an analogue, in terms of variables and substitution, for the higher-order operators of Section 2.1. It turns out that there is: one may present second-order operators as terms in *metavariable contexts* [Aczel 1978; Fiore and Hur 2010]. Formally, metavariables are variables that are themselves parameterised by variables: we can instantiate any metavariable by providing terms for each of its parameterising variables, akin to the application of (1) or substitution of (6). For example, the metavariable context below has n metavariables, each of which is parameterised by n_i variables.

$$(x_{1\,1}, \ldots, x_{1\,n_1})\mathrm{x}_1, \ldots, (x_{n\,1}, \ldots, x_{n\,n_n})\mathrm{x}_n \tag{7}$$

A second-order operator may be defined, similarly to (5), as a symbol parameterised by a context of metavariables, such as in the following inference rule:

$$\frac{}{(x_{1\,1}, \ldots, x_{1\,n_1})\mathrm{x}_1, \ldots, (x_{n\,1}, \ldots, x_{n\,n_n})\mathrm{x}_n \vdash \mathsf{g}} \tag{8}$$

We read this inference rule as "we may form a term g in any context with n metavariables, the i^{th} of which is parameterised by n_i variables". We understand g as some term containing free metavariables. Just as variables have an associated notion of substitution, metavariables have an associated notion of *meta-substitution* [Fiore 2008; Fiore and Hur 2010]. When we substitute a variable x by a term t, we replace every occurrence of the variable x by t (taking care to deal with binders appropriately); similarly, we may substitute a metavariable $(x_1, \ldots, x_n)\mathrm{x}$ by an open term $x_1, \ldots, x_n \vdash f$. This allows us to apply a second-order operator as in (8), by meta-substituting each of the metavariables $(x_{1\,1}, \ldots, x_{1\,n_1})\mathrm{x}_1$ through $(x_{n\,1}, \ldots, x_{n\,n_n})\mathrm{x}_n$ by open terms $x_{1\,1}, \ldots, x_{1\,n_1} \vdash f_1$

through to $x_{n\,1}, \ldots, x_{n\,n_n} \vdash f_n$, as below (we use the same notation for substitution and meta-substitution).

$$\frac{x_{1\,1}, \ldots, x_{1\,n_1} \vdash f_1 \qquad \cdots \qquad x_{n\,1}, \ldots, x_{n\,n_n} \vdash f_n}{\vdash \mathrm{g}[{}^{f_1}/_{x_1}] \cdots [{}^{f_n}/_{x_n}]} \tag{9}$$

Note that this inference rule has the same form as (3), under the relationship between the first-order operators exhibited by (1) and (6). As in the first-order setting, these two perspectives on second-order operators are equivalent. We may similarly describe third-order operators by way of metametavariables, metametasubstitution, and so on. In theory, we could combine the two perspectives, introducing metavariable contexts to the formalism of Section 2.1, but we gain no extra expressivity by doing so.

The perspective of equational logic with metavariables is well-suited to describing axiom schemata, which are typically formalised non-syntactically through an infinite family of axioms: metavariables permit them to be described syntactically, with each axiom of a schema arising from a higher-order operator by meta-substitution (cf. Fiore and Hamana [2013, Section 1]). In this sense, the notion of metavariable described here aligns with that of the traditional notion in mathematical logic. This is the perspective taken by Fiore and Hur [2010] in the setting of second-order equational logic (cf. Fiore [2008]; Hamana [2004]). In their setting, contexts contain both metavariables and variables. However, just as first-order operators are equivalent to nullary second-order operators, so variables are equivalent to nullary metavariables and so there is no loss in generality to consider solely contexts of metavariables.

2.3 Higher-order logical frameworks

Logical frameworks are deductive systems whose reasoning is expressed through a type theory, which plays the role of a metatheory. In particular, type theories with some capacity of function type form the metatheories for higher-order logical frameworks. Categorical theories (e.g. algebraic theories, essentially algebraic theories, geometric theories, etc.) have traditionally been studied separately from logical frameworks, but the objects of study are the same, albeit in different dress. For example, (multisorted) universal algebra can equivalently be viewed as the logical framework corresponding to the *simply-typed pairing calculus*: the fragment of the simply-typed λ-calculus with products but without function types (cf. Crole [1993, Chapter 3]); this view lends itself as a useful bridge between the approaches of categorical algebra and programming language theory.

Having made this observation, there is a clear candidate for the metatheory associated to higher-order equational logic: the simply-typed λ-calculus. Metavariables may be represented by variables of function types, while meta-substitution is given by the (ordinary) substitution of λ-terms. In fact, it is common in computer science to use the simply-typed λ-calculus to represent variable-binding operators, treating the λ-abstraction operator as a canonical variable-binding operator through which others may be defined: this is essentially the motivating idea behind higher-order universal algebra [Meinke 1992, 1995; Poigné 1986], and higher-order abstract syntax[1] [Pfenning and Elliott 1988]. However, one could argue that this practice was formally justified only once the binding structure of the simply-typed λ-calculus was proven to be universal, in the sense of being equivalent to arbitrary algebraic binding structure by Fiore and Mahmoud [2010]; Mahmoud [2011]. Following this result, we may in good conscience present n^{th}-order operators as operators with limited order in the simply-typed λ-calculus. For instance, we may present a second-order operator by a function

[1]We note that the metalogic of Pfenning and Elliott [1988] is also polymorphic, but reserve the term *higher-order abstract syntax* for the fragment restricted to the simply-typed λ-calculus.

constant, such as the following.

$$\frac{}{\vdash \mathrm{g} : (\mathsf{U}^{n_1} \to \mathsf{U}) \times \cdots \times (\mathsf{U}^{n_n} \to \mathsf{U}) \to \mathsf{U}} \tag{10}$$

Here, g is an operator taking functions as operands, and is equivalent to (8) by uncurrying. Given terms $\vdash f_1 : \mathsf{U}^{n_1} \to \mathsf{U}$ through to $\vdash f_n : \mathsf{U}^{n_n} \to \mathsf{U}$, corresponding to open terms by uncurrying, we may form a new term $\mathrm{g}(f_1, \ldots, f_n)$ using the application operation of the simply-typed λ-calculus:

$$\frac{\vdash f_1 : \mathsf{U}^{n_1} \to \mathsf{U} \qquad \cdots \qquad \vdash f_n : \mathsf{U}^{n_n} \to \mathsf{U}}{\vdash \mathrm{g}(f_1, \ldots, f_n) : \mathsf{U}} \tag{11}$$

Note that, though we distinguish informally between the operators defined using the simply-typed λ-calculus and the operators of the simply-typed λ-calculus itself, there is no formal difference between the two from this perspective.

The presentation of higher-order equational logic by the simply-typed λ-calculus is the one we choose to use throughout this paper, as the syntax is particularly elegant and is likely to be most familiar to the reader.

2.4 Simply-typed λ-calculi

We consider the simply-typed λ-calculus above as a logical framework for higher-order deduction. However, extensions of the simply-typed λ-calculus have often instead been studied for the purpose of defining programming languages. Conversely, in a logical framework, the primitive type and term operators have philosophical import: for instance, the product type corresponds to conjunction, and the function type to implication. In a programming language, they are concrete syntactic devices, and their meaning is defined through their behaviour. Practically, these perspectives are similar, but the distinction between taking the simply-typed λ-calculus as a metatheory, or as a programming language, is conceptually important.

Crole [1993, Chapter 4], for instance, takes a more programming-language-theoretic approach, defining a notion of $\lambda\times$*-theory* equivalent to our notion of presentation for an ω-order algebraic theory. Crole's motivation is to prove an internal language result for cartesian-closed categories, as well as to describe programming languages extending the simply-typed λ-calculus: correspondingly, he does not consider the presentation-free perspective, which is our primary motivation.

3 PRESENTATIONS OF HIGHER-ORDER ALGEBRAIC THEORIES

We proceed to describe an equational logic for higher-order algebraic theories, based on the perspective in Section 2.3, and define their corresponding presentations. This gives a concrete syntactic counterpoint to the later categorical formulation, allowing the higher-order equational logic to be used as an internal language for higher-order algebraic theories.

3.1 The order-limited λ-calculus

The classical correspondence between the simply-typed λ-calculus and cartesian-closed categories [Lambek 1980; Lambek and Scott 1988] is well-known: it establishes that simply-typed λ-calculi and cartesian-closed categories are equivalent notions, permitting us to treat each in terms of the other as convenient. The correspondence straightforwardly restricts to one between the simply-typed pairing calculus, the subcalculus of the simply-typed λ-calculus without function types, and cartesian categories. By restricting the simply-typed λ-calculus appropriately, we can establish that between these two lie a spectrum of calculi, ranging in expressivity, which we dub the *order-limited λ-calculi*. These will play the role of the metatheory, or equational logic, for n^{th}-order algebraic theories.

Remark 3.1. Order *is here used to refer to the order of functions, and by extension their types and calculi. The* second-order λ-calculus *refers to a simply-typed λ-calculus whose function types may be at most first-order, rather than the polymorphic λ-calculus, which has occasionally gone by that name.*

To describe the order-limited calculi, it shall be necessary to define the *order* of a type: informally the maximum left-nesting depth of any function type constructor therein. Some examples follow. The types of the simply-typed λ-calculus are generated from a set S of base types by the unit type (1), product types ($\times$), and function types ($\rightarrow$); throughout, B denotes some base type.

Order	Types			
0	1			
1	B	$\mathrm{B} \times \mathrm{B}$	$\mathrm{B} \times \mathrm{B} \times \mathrm{B}$	$\cdots$
2	$\mathrm{B} \rightarrow \mathrm{B}$	$\mathrm{B} \times \mathrm{B} \rightarrow \mathrm{B}$	$\mathrm{B} \rightarrow \mathrm{B} \times \mathrm{B}$	$\mathrm{B} \rightarrow \mathrm{B} \rightarrow \mathrm{B}$
3	$(\mathrm{B} \rightarrow \mathrm{B}) \rightarrow \mathrm{B}$	$\cdots$		

Definition 3.1. The *order* of a type in the simply-typed λ-calculus is given as follows.

$$\mathrm{ord}(1) \stackrel{\mathrm{def}}{=} 0 \qquad \mathrm{ord}(\mathrm{B}) \stackrel{\mathrm{def}}{=} 1 \quad (\mathrm{B} \in S)$$

$$\mathrm{ord}(X \times Y) \stackrel{\mathrm{def}}{=} \max(\mathrm{ord}(X), \mathrm{ord}(Y)) \qquad \mathrm{ord}(X \rightarrow Y) \stackrel{\mathrm{def}}{=} \max(\mathrm{ord}(X) + 1, \mathrm{ord}(Y))$$

For $n \in \mathbb{N}_\omega$, the $(n+1)^{\mathrm{th}}$-order simply-typed λ-calculus is given by the classical simply-typed λ-calculus, but whose types X are restricted to those for which $\mathrm{ord}(X) \leq n$. Consequently, the abstraction rule must be restricted, as in the following. The full calculus is presented in Figure A.1.

$$\frac{\Gamma, x : X \vdash t : Y \qquad \mathrm{ord}(X) < n}{\Gamma \vdash \lambda(x : X.t) : X \rightarrow Y} \ \rightarrow\text{-INTRO}$$

The 0^{th}-order simply-typed λ-calculus is restricted further, as the only terms therein are constant: it is presented in Figure A.2. In the following, we shall focus on $n > 0$. Note that the first-order simply-typed λ-calculus coincides with the simply-typed pairing calculus; while the ω-order simply-typed λ-calculus coincides with the classical simply-typed λ-calculus.

We may construct a category from the n^{th}-order simply-typed λ-calculus (cf. Crole [1993]).

Definition 3.2. Let $n \in \mathbb{N}_\omega$. The *classifying category* $\Lambda_n(S)$ of the n^{th}-order simply-typed λ-calculus on a set S of base types is the category defined as having

- objects, the types of the n^{th}-order simply-typed λ-calculus on S;
- morphisms $X \longrightarrow Y$, the terms $x : X \vdash t : Y$;
- identity morphisms $X \longrightarrow X$, variable projections $x : X \vdash x : X$;
- compositions $(x : X) \xrightarrow{s} (y : Y) \xrightarrow{t} Z$, substitutions $t[{}^{s}/_{y}]$.

The classifying categories of the simply-typed pairing calculus and of the simply-typed λ-calculus are characterised by universal properties: the former is the free cartesian category on S, while the latter is the free cartesian-closed category on S. One should hope for a similar characterisation of $\Lambda_{n+1}(S)$ for general $n \in \mathbb{N}_\omega$. In the $(n+1)^{\mathrm{th}}$-order simply-typed λ-calculus, there are restrictions on forming function types. One should therefore expect $\Lambda_{n+1}(S)$ to always have cartesian structure, but only limited closed structure. This limited closed structure is captured categorically by the notion of exponentiability.

Definition 3.3. An object X in a cartesian category is *exponentiable* iff the functor $X \times (-)$ has a right adjoint, typically denoted $(-)^X$.

In the context of the $(n+1)^{\text{th}}$-order simply-typed λ-calculus, a type is exponentiable if it may be used as the domain of a function type. To capture the notion of order, we shall be concerned with objects whose powers are exponentiable, corresponding to the left-nesting depth of function types.

Definition 3.4. We define *tetration* for an object X in a cartesian category inductively, whenever the requisite powers exist. Intuitively, this corresponds to iterated exponentiation of X.

$$X \upuparrows 0 \stackrel{\text{def}}{=} 1 \qquad\qquad X \upuparrows (n+1) \stackrel{\text{def}}{=} X^{X \upuparrows n} \qquad (n \in \mathbb{N})$$

An object is *n-tetrable*, for $n \in \mathbb{N}$, if for all $0 \leq i \leq n$ the object $X \upuparrows i$ is exponentiable. An object is *ω-tetrable* if it is n-tetrable for all $n \in \mathbb{N}$. It follows that every object in a cartesian category is 0-tetrable, and is 1-tetrable iff it is exponentiable. In a cartesian-closed category, every object is ω-tetrable, as is the terminal object in a cartesian category.

Intuitively, tetrability is inverse to order: a type in the $(n+1)^{\text{th}}$-order algebraic theory has order $1 \leq k \leq n$ if it is $(n+1-k)$-tetrable as an object of $\Lambda_{n+1}(S)$; the base types of the $(n+1)^{\text{th}}$-order simply-typed λ-calculus, having order 1, are therefore n-tetrable in $\Lambda_{n+1}(S)$. Just as types of a fixed order are closed under taking product types, n-tetrable objects are closed under taking cartesian products. Tetrability is therefore a faithful categorical reflection of order. The following definition captures the idea of a category equipped with base types of order 1.

Definition 3.5. A subcategory $\mathscr{C}'$ of a cartesian category $\mathscr{C}$ is *n-tetrable* iff each of the objects of $\mathscr{C}'$ is n-tetrable as an object of $\mathscr{C}$.

We are now ready to establish the universal property of $\Lambda_{n+1}(S)$.

Theorem 3.6. *Let $n \in \mathbb{N}_\omega$. $\Lambda_{n+1}(S)$ is the 2-initial cartesian category containing S as an n-tetrable subcategory. This exhibits $\Lambda_{n+1}(S)$ as the free cartesian category with an n-tetrable subcategory S.*

$\Lambda_1(S)$ and $\Lambda_\omega(S)$ are thereby the free cartesian and free cartesian-closed categories on S.

Remark 3.2. *Presenting the equational logic of higher-order algebraic theories as order-limited λ-calculi leads to several simplifications over previous approaches. For example, the meta-substitution operation of Fiore [2008] is given in our framework by the substitution of a second-order variable by a λ-abstraction. The near-semiring compatibility structure between substitution and meta-substitution observed by Fiore [2016] then follows directly from the associativity of substitution.*

3.2 Presentations, transliterations and translations

We may now describe the presentations of n^{th}-order algebraic theories for $0 < n \in \mathbb{N}_\omega$, which are analogous to those in universal algebra. Presentations allow higher-order algebraic theories to be axiomatised by means of operators and equations.

Definition 3.7. An S-sorted n^{th}-order *signature* consists of a set O of operators and a function $|-| : O \to \Lambda_n(S) \times S$. When $|o| = (X, B)$ we call X the *arity* and B the *coarity* of o. A signature gives rise to a *syntactic category* Λ_O defined as the classifying category in Definition 3.2 with the following additional axiom schema.

$$\frac{\Gamma \vdash t : X}{\Gamma \vdash \mathrm{o}(t) : \mathrm{B}} \ (\mathrm{o} \in O, |\mathrm{o}| = (X, \mathrm{B}))\ \textsc{op}$$

Λ_O is a wide subcategory of $\Lambda_n(S)$, which justifies us in treating their objects indiscriminately.

Definition 3.8. An S-sorted n^{th}-order *presentation* consists of a signature $(O, |-|)$ and a set $E \subseteq \sum_{(X,B) \in \Lambda_n(S) \times S} \Lambda_O(X, \mathrm{B}) \times \Lambda_O(X, \mathrm{B})$ of *equations*. Every presentation $\Sigma = (O, |-|, E)$ similarly gives rise to a syntactic category Λ_Σ defined as the syntactic category for the underlying signature with the following additional axiom schema.

$$\frac{\Gamma \vdash t : X}{\Gamma \vdash l[{}^{t}/_{x}] \equiv r[{}^{t}/_{x}] : \mathsf{B}} \ ((X, B, l, r) \in E)\ \textsc{eq}$$

We denote by $Q_\Sigma : \Lambda_O \twoheadrightarrow \Lambda_\Sigma$ the quotient of Λ_O by the equations of Σ.

There are natural notions of morphism between presentations: the first, which we call *transliterations*, are homomorphisms between signatures, mapping operators in one presentation to operators in another; the second, which we call *translations*[2] following Fiore and Mahmoud [2010], instead map operators in one presentation to *terms* in another. In practice, translations are more important, but we shall see shortly that the two notions are tightly connected. Morphisms of presentations are often overlooked in the literature, but are important both conceptually and practically: we give several examples in Section 3.3.

Definition 3.9. Let $\Sigma = (O, |{-}|, E)$ and $\Sigma' = (O', |{-}|', E')$ be n^{th}-order presentations. An n^{th}-order *transliteration* from Σ to Σ' consists of a function $f : O \to O'$ such that $|f(\mathrm{o})|' = |\mathrm{o}|$ for all $\mathrm{o} \in O$, and such that, for all $(\Gamma, \mathsf{B}) \in \Lambda_n(S) \times S$ and $l, r \in \Lambda_O(\Gamma, \mathsf{B})$, we have $Q_{\Sigma'}(f^\flat(l)) = Q_{\Sigma'}(f^\flat(r))$ if $Q_\Sigma(l) = Q_\Sigma(r)$, where $f^\flat$ is the congruent extension of f to terms.

S-sorted n^{th}-order presentations and transliterations form a category $\mathbf{Pre}_n(S)$, with composition and identities inherited from $\mathbf{Set}$.

Every transliteration $f : \Sigma \to \Sigma'$ gives rise to a *syntactic functor* $\Lambda_f : \Lambda_\Sigma \to \Lambda_{\Sigma'}$, defined as the identity on $\Lambda_n(S)$ and sending each term $\Gamma \vdash \mathrm{o}(t) : \mathsf{B}$ to $\Gamma \vdash f(\mathrm{o})(f^\flat(t)) : \mathsf{B}$, for all $\mathrm{o} \in O$.

Definition 3.10. Let $\Sigma = (O, |{-}|, E)$ and $\Sigma' = (O', |{-}|', E')$ be n^{th}-order presentations. An n^{th}-order *translation* from Σ to Σ' consists of a function $f : \prod_{\mathrm{o} \in O} \Lambda_{O'}(|\mathrm{o}|)$, such that, for all $(\Gamma, \mathsf{B}) \in \Lambda_n(S) \times S$ and $(l, r) \in \Lambda_O(\Gamma, \mathsf{B})$, we have $Q_{\Sigma'}(f^\sharp(l)) = Q_{\Sigma'}(f^\sharp(r))$ if $Q_\Sigma(l) = Q_\Sigma(r)$, where $f^\sharp$ is the congruent extension of f to terms.

S-sorted n^{th}-order presentations and translations form a category $\mathbf{Pres}_n(S)$, with identities given by inclusions, and compositions $g \circ f$ given by $g^\sharp \circ f$.

3.3 Examples

We give a range of examples of presentations and translations for higher-order algebraic theories.

Example 3.11. The *unityped λ-calculus* is a second-order algebraic theory presented by a single sort U together with the following operators and equations.

$$\frac{\Gamma \vdash f : \mathsf{U} \qquad \Gamma \vdash x : \mathsf{U}}{\Gamma \vdash \mathsf{app}(f, x) : \mathsf{U}}\ \mathsf{U}\text{-}\textsc{intro} \qquad \frac{\Gamma \vdash f : \mathsf{U} \to \mathsf{U}}{\Gamma \vdash \mathsf{abs}(f) : \mathsf{U}}\ \mathsf{U}\text{-}\textsc{elim}$$

$$\frac{\Gamma \vdash f : \mathsf{U} \to \mathsf{U} \qquad \Gamma \vdash u : \mathsf{U}}{\Gamma \vdash \mathsf{app}(\mathsf{abs}(f), u) \equiv f\,u : \mathsf{U}}\ \mathsf{U}\text{-}\beta$$

The unityped λ-calculus is called *extensional* when equipped with the U-η rule.

$$\frac{\Gamma \vdash f : \mathsf{U}}{\Gamma \vdash \mathsf{abs}(\lambda x : \mathsf{U}.\, \mathsf{app}(f, x)) \equiv f : \mathsf{U}}\ \mathsf{U}\text{-}\eta$$

The *continuation-passing style transform* forms a second-order translation from the unityped λ-calculus to itself [Mahmoud 2011, Example 6.2(3)].

[2] Lambek and Scott [1988] use *translation* to refer instead to mappings from *terms* to terms: the congruent extensions we define below are examples. The precise relationship between these two definitions is given in Section 5.1.

Example 3.12. The *simply-typed λ-calculus* on a set of base types S is an $\mathrm{ob}(\Lambda_\omega(S))$-sorted second-order algebraic theory, presented by the usual rules for the simply-typed λ-calculus (e.g. those for $n = \omega$ in Figure A.1). Note that this example demonstrates that we may express arbitrary higher-order structure in a second-order algebraic theory, but only given an infinite set of sorts.

Example 3.13. The natural numbers, with addition and multiplication, form a monosorted first-order algebraic theory. There is a second-order translation from the aforesaid theory of arithmetic to the unityped λ-calculus given by Church encoding [Mahmoud 2011, Example 6.2(2)].

Example 3.14. For all $n \in \mathbb{N}$, n^{th}-order logic is an $(n+1)^{\text{th}}$-order algebraic theory. Higher-order logic is an ω-order algebraic theory. Analogously, Hilbert's ϵ-calculus is a second-order algebraic theory, where the choice operator ϵ is second-order (cf. Escardó and Oliva [2010b]).

Example 3.15. Staton's *parameterised algebraic theories* [2013a; 2013b] are $\{\mathsf{P}, \mathsf{T}\}$-sorted[3] second-order algebraic theories whose binding operands have arity $\mathsf{P}^n \to \mathsf{T}$ for $0 < n \in \mathbb{N}$ and whose operations with coarity P are monosorted. Consequently, examples of parameterised algebraic theories, such as Fiore and Staton's theory of jumping [Fiore and Staton 2014], and the equational theory of the Beta-Bernoulli process [Staton, Stein, Yang, Ackerman, Freer, and Roy 2018], are also examples of second-order algebraic theories.

Example 3.16. Context-free expressions form a monosorted second-order algebraic theory, extending regular expressions with a least fixed-point operator μ [Krishnaswami and Yallop 2019].

Example 3.17. Plotkin's axiomatisation of partial differentiation [2020] is a monosorted second-order algebraic theory.

Example 3.18. Control operators are presented by two sorts $\{\mathsf{A}, \mathsf{Z}\}$ together with third-order operators, subject to various equations, typically forcing Z to be uninhabited. Examples include Felleisen and Friedman's control operator [Felleisen and Friedman 1986; Griffin 1989] and call/cc [Hofmann 1995], given respectively by the following inference rules.

$$\frac{\Gamma \vdash f : (\mathsf{A} \to \mathsf{Z}) \to \mathsf{Z}}{\Gamma \vdash C(f) : \mathsf{A}} \qquad \frac{\Gamma \vdash f : (\mathsf{A} \to \mathsf{Z}) \to \mathsf{A}}{\Gamma \vdash \mathsf{call/cc}(f) : \mathsf{A}}$$

There is a third-order translation from call/cc to C (described in Escardó and Oliva [2010a] as a monad morphism from the selection monad to the continuation monad) that maps call/cc to the term $f : (\mathsf{A} \to \mathsf{Z}) \to \mathsf{A} \vdash C(\lambda g.g\ (f\ g)) : \mathsf{A}$.

4 FREE CARTESIAN-CLOSED CATEGORIES

Having described presentations for higher-order algebraic theories, we shall proceed to describe the categorical formalism through which we investigate their structure.

To give context for the following development, we recall that an S-sorted (first-order) *algebraic theory* is a cartesian category $\mathscr{L}$ equipped with a strict cartesian identity-on-objects functor $L : \mathbf{Cart}(S) \to \mathscr{L}$, where $\mathbf{Cart}(S)$ is the free strict cartesian category on S. This definition may initially seem opaque, but we note that an S-sorted algebraic theory may equivalently be considered a cartesian category with specified finite products and a specified set of generators S, which is the same structure as described by the pairing calculus (or equivalently, by universal algebra). A higher-order algebraic theory should be considered similarly, but for which there is additionally specified exponentiable structure. To describe such structure, we require a higher-order replacement for $\mathbf{Cart}(S)$. Theorem 3.6 suggests such a replacement: the category $\Lambda_n(S)$. However, $\Lambda_n(S)$ is

[3] Multisorted parameterised algebraic theories, as introduced in Staton [2013b], may be similarly represented.

defined syntactically, which is unsatisfactory: we would prefer a direct, combinatorial construction. This preference could simply be justified on aesthetic grounds, but we will also find that a direct definition will be valuable for describing the structure of n^{th}-order algebraic theories.

4.1 Cartesian-closed categories of trees

Theorem 3.6 establishes the S-sorted n^{th}-order simply-typed λ-calculus as a syntactic characterisation of the free cartesian category with an n-tetrable subcategory S. However, as given, the description of $\Lambda_n(S)$ contains redundancies; we shall now proceed to prune these redundancies, leaving us with a direct definition.

Objects. Consider the types of $\Lambda_n(S)$, which are formed inductively from the base types $\mathrm{B} \in S$, the unit type, binary products types, and function types. The following isomorphisms, for all $X, Y, Z \in \Lambda_n(S)$, follow from the universal properties of finite products and exponential objects.

$$\begin{array}{lrlr} 1 \times X \cong X \cong X \times 1 & \times\ \textit{unit} & X \times (Y \times Z) \cong (X \times Y) \times Z & \times\ \textit{associativity} \\ X^1 \cong X & \rightarrow \textit{left-unit} & (Z^Y)^X \cong Z^{X \times Y} & \textit{currying} \\ 1^X \cong 1 & \rightarrow \textit{right-zero} & (Y \times Z)^X \cong Y^X \times Z^X & \times\ \textit{left-distributes over} \rightarrow \end{array}$$

Each type may be represented by an abstract syntax tree, whose branches are labelled with $\times$ and $\rightarrow$, and whose leaves are labelled by either elements of S or the type 1. However, due to the isomorphisms above, each type is represented (up to isomorphism) by many different trees. We describe a procedure to normalise a tree, producing a canonical representation.

First note that every binary product may either be eliminated by currying, or lifted to the root by left-distributing. Similarly, any unit type that is not at the root may be eliminated by the unit and zero isomorphisms. Since binary products are associative, we can equivalently consider lists of abstract syntax trees whose branches are labelled (trivially) by $\rightarrow$ and whose leaves are labelled by elements of S. Every object of $\Lambda_n(S)$ is described by such a list of trees, but to form a correspondence in both directions we must restrict the trees to limit the order of each type to less than n. The order of each type is given by the maximum number of left-steps in a path from the root of the corresponding tree to any leaf: note that each of the isomorphisms above preserves this property. We now make this intuition precise.

Definition 4.1. We denote by $\mathrm{Tree}(S) = \mu X.\, 2{\times}X + S$ the set of binary trees whose leaves are labelled by elements of S. The *left-width* of a binary tree is defined as the maximum number of left-steps from its root to any leaf, explicitly by the following function $\ell : \mathrm{Tree}(S) \longrightarrow \mathbb{N}$.

$$\ell(\iota_1(l, r)) = \max(1 + \ell(l), \ell(r)) \qquad\qquad \ell(\iota_2(s)) = 0$$

We then denote by $\mathrm{Tree}_n(S)$ the restriction of $\mathrm{Tree}(S)$ to those trees t such that $\ell(t) \leq n$, and by $\mathrm{Col}_n(S)$ (for *colonnade*: a row of trees) the set $\mathrm{List}(\mathrm{Tree}_n(S))$ of ordered lists of such trees.

Every object in $\Lambda_n(S)$ is isomorphic to one represented by an element of $\mathrm{Col}_n(S)$.

Morphisms. Consider the morphisms of $\Lambda_n(S)$, as given between elements of $\mathrm{Col}_n(S)$. To give a morphism into a list is to give a morphism into each of its elements. By uncurrying, to give a morphism into a tree is to give a morphism (with an extended domain) into its rightmost leaf. Therefore, it suffices to consider the structure of morphisms whose codomain is given by a base type B.

The terms in the n^{th}-order simply-typed λ-calculus, for a fixed context and type, form an equivalence class, according to the β- and η-laws. To pick a canonical inhabitant of each class is to define a set of normal-form terms. We will describe a procedure to enumerate the normal-form terms for the order-limited λ-calculi: every morphism in the corresponding syntactic category will therefore be uniquely described by a single canonical term.

Consider a term $x_1 : X_1, \ldots, x_k : X_k \vdash t : \mathsf{B}$, where each X_i is represented by a tree on S, and $\mathsf{B} \in S$ is a base type. The only way to form such a term t is to project a variable x_i whose type X_i has rightmost leaf B, and then to provide terms for each of its arguments. The procedure for producing a term for an argument $x : X$ is the same as that for producing the original term t, except that X may be a function type: in this case, we expand the context by a fresh variable whose type matches the domain of X. The two processes – recursively providing arguments, and expanding the context for function types – correspond to β-reduction and η-expansion respectively. Note that forming new unit, pair, or λ-terms during the procedure is never necessary: this inductive structure is entirely determined by the choices of variable projections from the context. This procedure produces the full set of β-short η-long normal-form terms for a given context and type.

Construction. In concrete terms, the descriptions of types as colonnades and terms in normal form gives an explicit presentation of the free cartesian category with an n-tetrable subcategory. We define the following inductive functions: intuitively, $\nu(\Gamma, X)$ is the set of $\beta\eta$-normal terms $\Gamma \vdash t : X$; $\upsilon(\Gamma, B_i)$ produces the uncurried form of a context and type; and $\rho(\Gamma, \mathsf{B})$ is the set of $\beta\eta$-normal terms $\Gamma \vdash t : \mathsf{B}$, for $\mathsf{B} \in S$ a base type. $+\!\!+$ denotes list concatenation.

$$\nu(\Gamma, X) = \prod_{B_i \in X} \rho(\upsilon(\Gamma, B_i))$$

$$\upsilon(\Gamma, B_i) = \begin{cases} \upsilon(\Gamma +\!\!+ [B'], B) & B_i = \varkappa_1(B', B) \\ (\Gamma, \mathsf{B}) & B_i = \varkappa_2 \mathsf{B} \end{cases} \qquad \rho(\Gamma, B) = \sum_{X_i \in \Gamma} \begin{cases} \nu(\Gamma +\!\!+ [X'], X) & X_i = \varkappa_1(X', X) \\ \{*\} & X_i = \varkappa_2 \mathsf{B} \\ \emptyset & X_i = \varkappa_2 \mathsf{B}',\ B' \neq \mathsf{B} \end{cases}$$

This construction straightforwardly generalises from a set of base types S to a small category $\mathbb{S}$ of base types and constants.

Definition 4.2. For $0 < n \in \mathbb{N}_\omega$, the *$n^{th}$-order theory of equality* $\mathbb{L}_n(S)$ on a set S is the category defined as having

- objects, the elements of $\mathrm{Col}_n(S)$;
- morphisms $\Gamma \longrightarrow X$, elements of $\nu(\Gamma, X)$;
- identity morphisms and compositions as in $\Lambda_n(S)$;

Proposition 4.3. $\mathbb{L}_n(S) \simeq \Lambda_n(S)$ □

The internal language of $\mathbb{L}_n(S)$ is a $\beta\eta$-normal order-limited simply-typed λ-calculus. In this sense, it may be viewed as a *lambda-free logical framework* [Adams 2008]: terms are given through parameterisation and instantiation, rather than abstraction and application. Consequently, the structure of $\mathbb{L}_n(S)$ lends itself to representations of higher-order abstract syntax. This characterisation is more convenient to analyse than the syntactic definition given by $\Lambda_n(S)$: for instance, we recover the usual characterisation of $\mathbf{Cart}(\mathbf{1})$ as a skeleton of $\mathbf{FinSet}$. In particular, unlike $\Lambda_n(S)$, the category $\mathbb{L}_n(S)$ satisfies a stronger, strict universal property.

Theorem 4.4. *$\mathbb{L}_{n+1}(S)$ is the initial strict cartesian category containing S as a strictly n-tetrable subcategory, for $n \in \mathbb{N}_\omega$.*

Fiore and Mahmoud [2010, Section 4] give an alternative syntactic description of $\mathbb{L}_2(1)$, which they call $\mathbf{M}$. Though it is presented differently, the universal property implies $\mathbb{L}_2(1) \cong \mathbf{M}$.

For $0 < m \leq n \in \mathbb{N}_\omega$ there is an inclusion $\mathbb{L}_m(S) \hookrightarrow \mathbb{L}_n(S)$, exhibiting $\mathbb{L}_m(S)$ as a (strictly) full subcategory of $\mathbb{L}_n(S)$. This in particular implies the following conservative extension result.

PROPOSITION 4.5. *The $(n+1)^{th}$-order simply-typed λ-calculus on S is a (faithful) conservative extension of the n^{th}-order simply-typed λ-calculus on S, for $n \in \mathbb{N}_\omega$.*

5 HIGHER-ORDER ALGEBRAIC THEORIES

We have now developed sufficient theory to present the main definitions of this paper. Higher-order algebraic theories provide a presentation-free formalism of higher-order structure. Any given algebraic structure may be described by numerous different presentations and it is therefore useful to have a unique representation that does not distinguish between these presentations. Just as (first-order) algebraic theories represent the structure of equational logic, so higher-order algebraic theories represent the structure of higher-order equational logic. We shall see that this categorical formalism allows us to prove powerful results about higher-order structure.

Definition 5.1. An *S-sorted n^{th}-order algebraic theory* is a cartesian category $\mathscr{L}$, equipped with a strict cartesian identity-on-objects functor $L : \mathbb{L}_n(S) \to \mathscr{L}$ strictly preserving the exponentiable objects. A *map* $F : L \to L'$ of S-sorted n^{th}-order algebraic theories is a functor $F : \mathscr{L} \to \mathscr{L}'$ such that $F \circ L = L'$. S-sorted n^{th}-order algebraic theories and their maps form a category $\mathbf{Law}_n(S)$.

The requirement that F commutes with L and L' equivalently means F strictly preserves the finite products and exponentiable objects.

5.1 Equivalence with presentations

We establish that the notions of presentation and theory we have given are equivalent, justifying their interchangeable use. This gives intuition for the definition of the latter: every n^{th}-order algebraic theory may be thought of as the classifying category for some (non-unique) n^{th}-order presentation. First, we clarify the relationship between transliterations and translations of presentations. The syntactic category and syntactic functor constructions of Section 3.2 extend to a functor $\Lambda_{(-)} : \mathbf{Pre}_n(S) \to \mathbf{Law}_n(S)$ given by taking the canonical inclusion of $\mathbb{L}_n(S) \simeq \Lambda_n(S)$ into the classifying category for each presentation. There is also a functor $\Pi_{(-)} : \mathbf{Law}_n(S) \to \mathbf{Pre}_n(S)$ in the other direction, which sends a theory $L : \mathbb{L}_n(S) \to \mathscr{L}$ to the signature whose operators are given by morphisms of $\mathscr{L}$. These functors form an adjunction $\Lambda_{(-)} \dashv \Pi_{(-)}$. Translations are exactly morphisms in the Kleisli category of the induced monad $\Pi_{(-)} \circ \Lambda_{(-)}$. However, $\Lambda_{(-)}$ and $\Pi_{(-)}$ themselves already form a Kleisli adjunction, since $\Lambda_{(-)}$ is essentially surjective: $\mathbf{Law}_n(S)$ is hence equivalent to the Kleisli category, giving us the desired equivalence with presentations.

THEOREM 5.2. $\mathbf{Law}_n(S) \simeq \mathbf{Pres}_n(S)$.

It is natural to also look at the Eilenberg–Moore category for the monad induced by this adjunction. In fact, $\Pi_{(-)}$ is fully faithful, so the Eilenberg–Moore category too is equivalent to $\mathbf{Law}_n(S)$. Intuitively, the characterisation of $\mathbf{Law}_n(S)$ as a Kleisli category corresponds to the presentation of a type theory by operators and equations (as in the approach of Crole [1993, Discussion 4.9.6]); and the characterisation as an Eilenberg–Moore category to its equivalent presentation as a set of terms closed under the deductive operations (as in the approach of Lambek and Scott [1988, Section I.10]).

REMARK 5.1. *The characterisation of $\mathbf{Pres}_1(S)$ as a Kleisli category was first observed in Vidal and Tur [2010, Proposition 5.11]. The equivalence $\mathbf{Pres}_2(1) \simeq \mathbf{Law}_2(1)$ was established in Mahmoud [2011, Theorem 6.6]. The observations that the Kleisli characterisation permits a proof of the equivalence, and that the Kleisli category is equivalent to the Eilenberg–Moore category, appear to be new.*

5.2 Coreflections between categories of theories

The equivalence between theories and presentations may be used to show that $\mathbf{Law}_n(S)$ is a coreflective subcategory of $\mathbf{Law}_{n+1}(S)$. In other words, there is a fully faithful functor $\ulcorner - \urcorner : \mathbf{Law}_n(S) \to$

$\mathbf{Law}_{n+1}(S)$, and this has a right adjoint $\lfloor - \rfloor : \mathbf{Law}_{n+1}(S) \to \mathbf{Law}_n(S)$. Informally, $\lceil - \rceil$ freely adds the $(n+1)^{\text{th}}$-order arities, and their associated evaluation and projection morphisms, to an n^{th}-order theory, while $\lfloor - \rfloor$ forgets the $(n+1)^{\text{th}}$-order structure. The counit $\lceil \lfloor L \rfloor \rceil \to L$ of the adjunction maps the freely added $(n+1)^{\text{th}}$-order structure to the same structure in L; while the unit $L \to \lfloor \lceil L \rceil \rfloor$ is an isomorphism, since the structure that is forgotten is the structure that was freely added. These coreflections are crucial to our development: for instance, they are used to define the notion of term algebra (Section 5.4), and to develop the monad–theory correspondence (Section 6 to Section 8).

We consider here the case $0 < n \in \mathbb{N}_\omega$; the case $n = 0$, with $\mathbf{Law}_0(S) = \mathbf{Set}^S$, requires separate treatment but presents no particular difficulty. Every n^{th}-order presentation can be considered as an $(n+1)^{\text{th}}$-order presentation, with only n^{th}-order operators and equations. The left adjoint $\lceil - \rceil$ is the composition of this embedding with the equivalences between theories and presentations:

$$\lceil - \rceil : \mathbf{Law}_n(S) \xrightarrow{\simeq} \mathbf{Pres}_n(S) \hookrightarrow \mathbf{Pres}_{n+1}(S) \xrightarrow{\simeq} \mathbf{Law}_{n+1}(S)$$

For the right adjoint $\lfloor - \rfloor$, we generalise a construction of Fiore and Mahmoud [2010]. Given an $(n+1)^{\text{th}}$-order algebraic theory $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$, define $\lfloor \mathscr{L} \rfloor$ to be the full subcategory of $\mathscr{L}$ on $\mathbb{L}_n(S)$. Explicitly, $\lfloor \mathscr{L} \rfloor$ has the same objects as $\mathbb{L}_n(S)$ and hom-sets $\lfloor \mathscr{L} \rfloor(X, Y) = \mathscr{L}(X, Y)$. The n^{th}-order algebraic theory $\lfloor L \rfloor$ is then given by the identity-on-objects functor $\lfloor L \rfloor : \mathbb{L}_n(S) \to \lfloor \mathscr{L} \rfloor$ defined on morphisms as $\lfloor L \rfloor(f) = L(f)$.

Theorem 5.3. *For each $n \in \mathbb{N}_\omega$, the constructions above form an adjunction,*

$$\begin{tikzcd} \mathbf{Law}_n(S) \arrow[r, shift left=1.5, "\lceil - \rceil"] \arrow[r, phantom, "\perp"] & \mathbf{Law}_{n+1}(S) \arrow[l, shift left=1.5, "\lfloor - \rfloor"] \end{tikzcd}$$

with $\lceil - \rceil$ fully faithful. Hence $\mathbf{Law}_n(S)$ is a coreflective subcategory of $\mathbf{Law}_{n+1}(S)$.

As an aside, we remark that coreflectivity implies that n^{th}-order algebraic theories are coalgebras for the idempotent comonad induced by this adjunction.

Coreflections compose, and since a similar construction demonstrates that each $\mathbf{Law}_n(S)$ is coreflective in $\mathbf{Law}_\omega(S)$, $\mathbf{Law}_m(S)$ is therefore a coreflective subcategory of $\mathbf{Law}_n(S)$ for all $m \leq n \in \mathbb{N}_\omega$.

As might be supposed, $\mathbf{Law}_\omega(S)$ is the limit of the coreflections in an appropriate sense. (Note, however, that it is not the colimit of the inclusions: this is instead given by a category of higher-order algebraic theories, such that, for each theory L, there exists natural number $k \in \mathbb{N}$ such that L is k^{th}-order.)

Proposition 5.4. *$\mathbf{Law}_\omega(S)$ is the limit of the ω-cochain $\mathbf{Law}_1(S) \xleftarrow{\lfloor - \rfloor} \mathbf{Law}_2(S) \xleftarrow{\lfloor - \rfloor} \cdots$ in $\mathbf{Cat}$.*

5.3 Models and strict models

The equivalence between theories and presentations exhibits higher-order algebraic theories as syntactic: we now consider the appropriate notion of semantics. The structure of a higher-order algebraic theory may be interpreted in another category, allowing us to reason about higher-order structure in various settings. For instance, a monoid is traditionally modelled by any set equipped with a unital associative binary operation. However, restricting to **Set** is unnecessary: one can just as readily model a monoid in any cartesian category. The same is true for higher-order algebraic theories: here we require sufficient exponentiable structure to interpret the higher-order operations.

Definition 5.5. A *model* for an S-sorted n^{th}-order algebraic theory $L : \mathbb{L}_n(S) \to \mathscr{L}$ in a cartesian category $\mathscr{C}$ is a cartesian functor $M : \mathscr{L} \to \mathscr{C}$ preserving the exponentiable objects. A *map* of models from M to M' is a natural transformation $M \Rightarrow M'$. Models for $\mathscr{L}$ and their maps form a category $\mathbf{Mod}(L, \mathscr{C})$, functorial contravariantly in the first argument and covariantly in the second.

Remark 5.2. *We caution that our definition of model is not the same as that of Fiore and Mahmoud [2010]; Mahmoud [2011]: their notion is equivalent to what we call a* term algebra *(Definition 5.12).*

This definition is a manifestation of Lawvere's *functorial semantics* [1963], in which models are given by structure-preserving functors from a theory. Note that natural transformations between functors preserving cartesian and exponentiable structure automatically preserve cartesian and exponentiable structure themselves, which justifies their role as maps of models.

Example 5.6. Models of the unityped λ-calculus (Example 3.11) (with U-η) in a category $\mathscr{C}$ are equivalently (extensional) reflexive objects in $\mathscr{C}$, i.e. exponentiable objects U equipped with an isomorphism $U \cong U^U$. Note that the only reflexive object in **Set** is the terminal object: preservation of exponentials is a strong requirement, especially for monosorted theories. In a sense, the unityped λ-calculus is the minimal monosorted higher-order algebraic theory and, as such, we should not expect to find many set-theoretic models of monosorted higher-order algebraic theories.

A model of the theory of simply-typed λ-calculus (Example 3.12) with base types S, in a cartesian-closed category $\mathscr{C}$, is equivalently an interpretation of the base types as objects of $\mathscr{C}$.

One particularly important class of models of L is given by the coslices under L, i.e. pairs of a theory L' and a map $L \to L'$. Each of these coslices is a model of L, since maps of n^{th}-order algebraic theories are structure-preserving. In some sense, the coslices are the canonical models for theories, in that the most natural structure in which to interpret a theory is precisely another theory, since both are equipped with the same fundamental deductive structure. Coslices may be considered the *strict* or *syntactic models* (insofar as theories themselves may be considered syntactic).

It is illuminating to explore the relationship between coslices and general models in a little more detail, but we can't immediately do so, because the categories of models assume a fixed codomain $\mathscr{C}$, whereas the codomains of the coslices vary. To consider the categories of models for arbitrary cartesian categories collectively, we will take the Grothendieck construction of $\mathbf{Mod}(L, -)$, which coherently combines each category of models. It is then clear that models strictly subsume coslices.

Proposition 5.7. *Let $L : \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory, and let $U : \mathbf{Law}_n(S) \to$ **Cart** be the functor forgetting the generating sorts and specified structure. The coslice category $L/\mathbf{Law}_n(S)$ is a non-full subcategory of $\int \mathbf{Mod}(L, U(-))$.*

Consequently, even when we consider models for L taken in other n^{th}-order algebraic theories, models are more general than coslices, since models are not required to strictly preserve the structure. We shall make one more edifying observation.

Proposition 5.8. *$\int \mathbf{Mod}(L, -)$ is equivalent to the subcategory of the lax coslice $\mathscr{L} /\!/ \mathbf{Cart}$ for which the coslices preserve exponentials.*

In this light, we may see models for an n^{th}-order algebraic theory L to correspond to a notion of coslice for a weaker notion of n^{th}-order algebraic theories, without specified structure: L is implicitly considered a weak n^{th}-order algebraic theory by forgetting the generating structure.

Many of the results about models in the first-order setting extend to the higher-order setting. For instance, one can freely construct strict models of n^{th}-order algebraic theories on lower-order structure. Since $\mathbf{Law}_0(S) = \mathbf{Set}^S$, we recover the classical free model construction on a set of constants, given by adjoining the constants to the theory.

Theorem 5.9. *Let $n \in \mathbb{N}_\omega$. The forgetful functor $\lfloor - \rfloor \circ U : L/\mathbf{Law}_{n+1}(S) \to \mathbf{Law}_{n+1}(S) \to \mathbf{Law}_n(S)$ has a left adjoint, sending L' to $L + \lceil L' \rceil$.*

Functors between categories of strict models induced by maps of theories have left adjoints, allowing us to freely construct models of one theory from another. Classically, this result is stated for fixed codomain; here we only prove the strict version.

Theorem 5.10. *Let $F : L \to L'$ be a map of S-sorted n^{th}-order algebraic theories. The functor $L'/\mathbf{Law}_n(S) \to L/\mathbf{Law}_n(S)$ taking a strict model for L' to its precomposition by F has a left adjoint.*

We note in passing that we can to some extent reconstruct a theory from a model (cf. [Power 2004, Proposition 1]), though we shall not explore the implications of this construction here.

Proposition 5.11. *Let $L : \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. For every model $M : \mathscr{L} \to \mathscr{C}$, there exists an n^{th}-order algebraic theory $L_M : \mathbb{L}_n(S) \to \mathscr{L}_M$, a map $F_M : L \to L_M$ and a fully faithful functor $M' : \mathscr{L}_M \to \mathscr{C}$ such that $M' \circ F_M \cong M$.*

5.4 Term algebras

For a first-order theory $L : \mathbb{L}_1(S) \to \mathscr{L}$, the terms in the empty context form a model $\mathscr{L}(1, -) : \mathscr{L} \to \mathbf{Set}$. For higher-order theories, this is no longer the case, since $\mathscr{L}(1, -)$ does not preserve exponentials. However, the structure formed by the constant terms of a theory is nevertheless important, and so we define a separate notion, that of *term algebras*, to capture it. Terms in the empty context of a theory form the prototypical example.

Term algebras intuitively describe the substitution structure of a theory. Before giving the definition, we shall consider the unityped λ-calculus (Example 3.11) as an exemplar. The closed terms of the theory $L_\lambda : \mathbb{L}_2(\{\mathsf{U}\}) \to \mathscr{L}_\lambda$ are given by the hom-sets $\mathscr{L}_\lambda(1, \mathsf{U}^{\mathsf{U}^n}) \cong \mathrm{Lam}_n$, where Lam_n is the set of open unityped λ-calculus terms with at most n free variables, up to $\beta\eta$-equality. The sets Lam_n are equipped with canonical substitution structure, and so we may assemble them into a category $\mathscr{L}_{\mathrm{Lam}}$ with objects $\mathsf{U}^n \in \mathbb{L}_1(\{\mathsf{U}\})$ and hom-sets $\mathscr{L}_{\mathrm{Lam}}(\mathsf{U}^n, \mathsf{U}^m) = {\mathrm{Lam}_n}^m$, where identities and composition are given by the variables and substitution respectively. In fact, this construction forms a first-order algebraic theory $L_{\mathrm{Lam}} : \mathbb{L}_1(\{\mathsf{U}\}) \to \mathscr{L}_{\mathrm{Lam}}$. Furthermore, the sets are equipped with interpretations of the λ-abstraction and application operators presented by L_λ, of the following form.

$$[\![\mathrm{abs}]\!] : \mathrm{Lam}_{n+1} \to \mathrm{Lam}_n \qquad [\![\mathrm{app}]\!] : \mathrm{Lam}_n \times \mathrm{Lam}_n \to \mathrm{Lam}_n \qquad (n \in \mathbb{N})$$

The first-order algebraic theory L_{Lam} induces a functor

$$\mathbb{L}_2(S) \xrightarrow{\lceil L_{\mathrm{Lam}} \rceil} \lceil \mathscr{L}_{\mathrm{Lam}} \rceil \xrightarrow{\lceil \mathscr{L}_{\mathrm{Lam}} \rceil(1,-)} \mathbf{Set}$$

The functions $[\![\mathrm{abs}]\!]$ and $[\![\mathrm{app}]\!]$ provide exactly the structure required to extend this functor to one $\mathscr{L}_\lambda \to \mathbf{Set}$. In general, the closed terms for an $(n+1)^{\mathrm{th}}$-order algebraic theory $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ form an n^{th}-order algebraic theory L', with $\mathscr{L}'(X, Y) = \mathscr{L}(1, Y^X)$, and the induced functor $\lceil \mathscr{L}' \rceil(1, \lceil L' \rceil(-)) : \mathbb{L}_{n+1}(S) \to \mathbf{Set}$ extends to a functor $\mathscr{L} \to \mathbf{Set}$. The extension interprets the operators of L as functions on closed terms. Term algebras axiomatise this situation, which describes the substitution structure of the closed terms of a theory.

Definition 5.12. Let $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{\mathrm{th}}$-order algebraic theory. The category L-$\mathbf{TmAlg}$ of *term algebras* for L is defined (up to isomorphism) by the following strict pullback in $\mathbf{CAT}$.

$$\begin{tikzcd}
L\text{-}\mathbf{TmAlg} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathscr{L}, \mathbf{Set}]} \arrow[d, "- \circ L"] \\
\mathbf{Law}_n(S) \arrow[r, "L' \mapsto \lceil \mathscr{L}' \rceil(1, \lceil L' \rceil(-))"'] & {[\mathbb{L}_{n+1}(S), \mathbf{Set}]}
\end{tikzcd}$$

Similar pullbacks are often used to define *models* of theories. This is because the notions of model and of term algebra have historically been conflated. We discuss the distinction between the two notions in more detail below. However, we note that first-order term algebras coincide with first-order models in **Set**. It can therefore be difficult to tell whether results in the first-order setting should apply to models or to term algebras when generalising to other settings.

Example 5.13. A term algebra for the algebraic theory L_λ of the unityped λ-calculus (Example 3.11) consists of a first-order algebraic theory $L : \mathbb{L}_1(S) \to \mathscr{L}$ and two families of functions,

$$[\![\mathsf{abs}]\!]_X : \mathscr{L}(X \times \mathsf{U}, \mathsf{U}) \to \mathscr{L}(X, \mathsf{U}) \qquad [\![\mathsf{app}]\!]_X : \mathscr{L}(X, \mathsf{U}) \times \mathscr{L}(X, \mathsf{U}) \to \mathscr{L}(X, \mathsf{U})$$

natural in $X \in \mathscr{L}$ and satisfying two equations, corresponding to the β- and η-laws respectively:

$$[\![\mathsf{app}]\!]_X([\![\mathsf{abs}]\!]_X(f), g) = f \circ \langle \mathrm{id}_X, g\rangle \qquad [\![\mathsf{abs}]\!]_X([\![\mathsf{app}]\!]_{X\times\mathsf{U}}(f \circ \pi_X, \pi_\mathsf{U})) = f$$

These are the λ*-theories* (with η), or *algebraic theories equipped with closed structure*, of Hyland [2017, Definition 3.1]. The (first-order) term algebras for the underlying first-order algebraic theory of the initial (second-order) term algebra for L_λ are Λ*-algebras* [Hyland 2017, Definition 4.1].

Term algebras for monosorted second-order algebraic theories are referred to as *models* in Fiore and Mahmoud [2010]; Mahmoud [2011].

The structure of the pullback defines a *semantics functor* $\mathrm{TmAlg} : \mathbf{Law}_{n+1}(S)^{\mathrm{op}} \to \mathbf{CAT}/\mathbf{Law}_n(S)$, which sends $L \in \mathbf{Law}_{n+1}(S)$ to the forgetful functor $L\text{-}\mathbf{TmAlg} \to \mathbf{Law}_n(S)$. Each map $L' \to L$ of $(n+1)^{\mathrm{th}}$-order algebraic theories induces a functor $L\text{-}\mathbf{TmAlg} \to L'\text{-}\mathbf{TmAlg}$ that commutes with the forgetful functors by the universal property of the pullback. One may use the semantics functor to show that theories are characterised by their categories of term algebras.

Lemma 5.14. *For each $n \in \mathbb{N}_\omega$, the semantics functor* $\mathrm{TmAlg} : \mathbf{Law}_{n+1}(S)^{\mathrm{op}} \to \mathbf{CAT}/\mathbf{Law}_n(S)$ *is fully faithful. In particular, it reflects isomorphisms.*

Term algebras have an equivalent characterisation as cartesian functors into **Set**. Let $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{\mathrm{th}}$-order algebraic theory. Note that the image of the projection $L\text{-}\mathbf{TmAlg} \to [\mathscr{L}, \mathbf{Set}]$ in the pullback above contains only cartesian functors. Conversely, given a cartesian functor $A : \mathscr{L} \to \mathbf{Set}$, we may construct an n^{th}-order algebraic theory L_A. For $n = 0$, we take the sorted set $L_A = (A(s))_{s\in S}$. For $n > 0$, we define $L_A : \mathbb{L}_n(S) \to \mathscr{L}_A$ as follows. The category $\mathscr{L}_A$ has hom-sets $\mathscr{L}_A(X, Y) = A(Y^X)$. Composition in $\mathscr{L}_A$ uses that A preserves products:

$$\mathscr{L}_A(Y, Z) \times \mathscr{L}_A(X, Y) = A(Z^Y) \times A(Y^X) \xrightarrow{\cong} A(Z^Y \times Y^X) \to A(Z^X) = \mathscr{L}_A(X, Z)$$

Products in $\mathscr{L}_A$ again use preservation of products, while exponentials are trivial:

$$\mathscr{L}_A(X, Z^Y) = A((Z^Y)^X) = A(Z^{X\times Y}) = \mathscr{L}_A(X \times Y, Z)$$

The identity-on-objects functor L_A is given on morphisms $f \in \mathbb{L}_n(S)(X, Y) \subseteq \mathbb{L}_{n+1}(S)(X, Y)$ by

$$\mathbb{L}_n(S)(X, Y) \xrightarrow{\cong} \mathbb{L}_{n+1}(S)(X, Y) \times 1 \xrightarrow{\cong} \mathbb{L}_{n+1}(S)(1, Y^X) \times A(1) \xrightarrow{(f,a)\mapsto A(L(f))(a)} A(Y^X) = \mathscr{L}_A(X, Y)$$

The theory $L_A \in \mathbf{Law}_n(S)$ is characterised by a natural isomorphism $\lceil \mathscr{L}_A \rceil(1, \lceil L_A \rceil(-)) \cong A \circ L$, analogous to commutativity of the pullback square above. The pair (L_A, A) induces an object of L-**TmAlg**, giving us the other direction of the equivalence.

Proposition 5.15. *Let $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{th}$-order algebraic theory, where $n \in \mathbb{N}_\omega$. Then* L-**TmAlg** *is equivalent to* $\mathbf{Cart}(\mathscr{L}, \mathbf{Set})$*, and this equivalence commutes up to natural isomorphism with the forgetful functors into* $\mathbf{Law}_n(S)$.

In particular, for first-order theories, we have $L\text{-}\mathbf{TmAlg} \simeq \mathbf{Mod}(L, \mathbf{Set})$. This coincidence is peculiar to first-order algebraic theories and we suggest it is responsible for the asymmetry between the behaviour of first-order models in **Set** compared to models in other cartesian categories. For instance, first-order algebraic theories correspond to certain monads on **Set** [Linton 1969], while the algebras for a monad induced by a theory correspond to models of the theory in **Set**. This is not the case for other cartesian categories. We argue this is because the monad correspondence should relate monad algebras and *term algebras*, rather than monad algebras and models. This distinction is clear in the higher-order setting, where models are not a generalisation of term algebras.

Let us give some intuition for this equivalent definition as cartesian functors. First, we note that the prototypical examples of term algebras, namely of terms in empty contexts, are captured.

Proposition 5.16. *Let $L : \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. Up to the equivalence of Proposition 5.15, the hom-functor $\mathscr{L}(1, -) : \mathscr{L} \to \mathbf{Set}$ is the initial term algebra.*

The initial term algebra for an n^{th}-order algebraic theory $L : \mathbb{L}_n(S) \to \mathscr{L}$ is therefore given by the sets of closed terms in L. The fact that the closed terms form a set is precisely the reason **Set** is distinguished amongst cartesian categories (it is the enriching category). Let $A : \mathscr{L} \to \mathbf{Set}$ be a cartesian functor. Each object $\Gamma \in \mathscr{L}$ may be considered a context $x_1 : X_1, \ldots, x_k : X_k$ in $\Lambda_n(S)$, and in this light one may think of the set $A(\Gamma)$ as being the set of terms that may be substituted for the variables x_1 through x_k. To provide a substitute for the entire context Γ is to provide a substitute for each variable x_i, and it is this that necessitates A be cartesian. Functoriality of A ensures that the substitutes are closed under the operations of L. Consequently, the substitution structure captured by Definition 5.12 is equivalently captured by cartesian functors into **Set**. In general, the substitutes in $A(\Gamma)$ are not terms in L; however, there is a way in which a term algebra can always be seen as being given by the sets of closed terms of *some* theory, which justifies our nomenclature.

Lemma 5.17. *Let $L : \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. For every term algebra $A : \mathscr{L} \to$ **Set**, there exists an n^{th}-order algebraic theory $L_A : \mathbb{L}_n(S) \to \mathscr{L}_A$ and a map $F_A : L \to L_A$ such that $\mathscr{L}_A(1, F_A(-)) \cong A$.*

In fact, this construction shows that term algebras and strict models are closely related.

Proposition 5.18. *L-**TmAlg** is a coreflective subcategory of $L/\mathbf{Law}_{n+1}(S)$, for $n \in \mathbb{N}_\omega$.*

Lemma 5.17 defines a functor $L_{(-)} : L\text{-}\mathbf{TmAlg} \simeq \mathbf{Cart}(\mathscr{L}, \mathbf{Set}) \to L/\mathbf{Law}_n(S)$ establishing every term algebra for $L : \mathbb{L}_n(S) \to \mathscr{L}$ as the initial algebra for some coslice under L. In particular, every term algebra can be seen as arising from the closed terms of some strict model of L. This construction has a right adjoint by Proposition 5.18: for every term algebra, we can construct a model whose closed terms coincide with those given by the term algebra. For finite n, this relationship is not an equivalence, because the reconstruction cannot recover the highest-order structure: given two objects X and Y, such that X is not exponentiable, we cannot recover morphisms $X \to Y$ by considering the closed terms $1 \to Y^X$. Therefore, there may be many coslices with isomorphic initial term algebras. When $n = \omega$, there is no such restriction and an equivalence holds.

Corollary 5.19. *For every $L : \mathbb{L}_\omega(S) \to \mathscr{L}$, there is an equivalence $L/\mathbf{Law}_\omega(S) \simeq L$-**TmAlg**.*

Alternatively, by freely adding exponentials with $\lceil - \rceil$, every object is made exponentiable, leading to the following result, which demonstrates that every strict n^{th}-order model is an $(n+1)^{\text{th}}$-order term algebra.

Proposition 5.20. *$L/\mathbf{Law}_n(S)$ is a coreflective subcategory of $\lceil L \rceil$-**TmAlg**, for $n \in \mathbb{N}_\omega$.*

For finite n, this is again not an equivalence: the action of a term algebra for $\lceil L \rceil$ on $(n+1)^{\text{th}}$-order morphisms cannot be recovered from an n^{th}-order model.

Term algebras play an important role in the subsequent monad–theory correspondence (Section 6 to Section 8) and the two perspectives, as collections of structured substitutes for the theory, or as the closed terms for some model of the theory, are helpful intuitions to keep in mind.

Remark 5.3. *While we suggest term algebras and models should be considered as fundamentally different, Garner [2014] proposes a method to extend the monad–theory correspondence to monads on arbitrary cartesian categories; in our language, this permits term algebras to be taken in categories other than* **Set**. *We leave understanding the precise relationship between models and term algebras in this enriched setting for future work.*

5.5 Theories from arities

We now show that each $(n+1)^{\text{th}}$-order arity $X \in \mathbb{L}_{n+1}(S)$ induces an n^{th}-order algebraic theory $p(X) \in \mathbf{Law}_n(S)$. We use this to demonstrate that $\mathbf{Law}_n(S)$ is locally strongly finitely presentable and to establish the monad–theory correspondence. In particular, the theories $p(X)$ form a generating set of "finite objects" in $\mathbf{Law}_n(S)$, and the monads induced by higher-order algebraic theories are determined by their actions on the theories $p(X)$.

Intuitively, the theories $p(X)$ are those that are presented by a finite number of operators and no equations. Let us consider an example. The object $(\mathsf{U} \times \mathsf{U} \to \mathsf{U}) \times ((\mathsf{U} \to \mathsf{U}) \to \mathsf{U})$ of $\mathbb{L}_3(\{\mathsf{U}\})$ induces an equation-free second-order algebraic theory with two operators, corresponding to the abstraction and application operators of the unityped λ-calculus as in Example 3.11. By the description of $\mathbb{L}_n(S)$ in Section 4.1, we may consider this second-order algebraic theory as described by the second-order λ-calculus with a constant of type $(\mathsf{U} \times \mathsf{U} \to \mathsf{U}) \times ((\mathsf{U} \to \mathsf{U}) \to \mathsf{U})$.

To define $p(X)$ formally, we use *simple slices* of theories. Simple slices of categories have previously proven useful in categorical treatments of type theory (e.g. Lambek and Scott [1988, Section I.7]); we take the name from Jacobs [1999, Definition 1.3.1]. Intuitively, the simple slice of L over X represents the extension of the theory L by a constant of type X.

Definition 5.21. Let $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{\text{th}}$-order algebraic theory and let $X \in \mathbb{L}_{n+1}(S)$ be an arity. We define the category $\mathscr{L}/\!\!/X$ to be the *simple slice* of $\mathscr{L}$ over X: the objects are the same as $\mathbb{L}_{n+1}(S)$; the hom-sets are defined $(\mathscr{L}/\!\!/X)(Y, Y') = \mathscr{L}(X \times Y, Y')$; identities are given by projections; and the composition of $g \in \mathscr{L}(X \times Y', Y'')$ with $f \in \mathscr{L}(X \times Y, Y')$ is $g \circ \langle \pi_X, f \rangle$. ($\mathscr{L}/\!\!/X$ is equivalently the Kleisli category for the comonad $X \times (-)$.) We define $L/\!\!/X : \mathbb{L}_{n+1}(S) \to \mathscr{L}/\!\!/X$ as the identity-on-objects functor sending $f \in \mathbb{L}_{n+1}(S)(Y, Y')$ to $L(f) \circ \pi_Y$, and call $L/\!\!/X \in \mathbf{Law}_{n+1}(S)$ the *simple slice* of L over X. This construction extends to a functor $L/\!\!/- : \mathscr{L}^{\text{op}} \to \mathbf{Law}_{n+1}(S)$.

We define a functor $p : \mathbb{L}_{n+1}(S)^{\text{op}} \to \mathbf{Law}_n(S)$ as the following, using the coreflection of $\mathbf{Law}_n(S)$ in $\mathbf{Law}_{n+1}(S)$ (Theorem 5.3), and the simple slice of the theory of equality $\mathrm{Id}_{\mathbb{L}_{n+1}(S)} \in \mathbf{Law}_{n+1}(S)$.

$$p : \mathbb{L}_{n+1}(S)^{\text{op}} \xrightarrow{\mathrm{Id}/\!\!/-} \mathbf{Law}_{n+1}(S) \xrightarrow{\lfloor - \rfloor} \mathbf{Law}_n(S)$$

The counit of the adjunction $\lceil - \rceil \dashv \lfloor - \rfloor$ is invertible on $\mathrm{Id}/\!\!/X$ for each $X \in \mathbb{L}_{n+1}(S)$ and so $\pi_X \in (\mathrm{Id}/\!\!/X)(1, X)$ induces a morphism $\rho_X \in \lceil p(X) \rceil(1, X)$. We then have the following universal property, which will be useful in reasoning about p.

Lemma 5.22. *Let $n \in \mathbb{N}_\omega$ and $X \in \mathbb{L}_{n+1}(S)$. For each n^{th}-order algebraic theory $L : \mathbb{L}_n(S) \to \mathscr{L}$ and $x \in \lceil \mathscr{L} \rceil(1, X)$, there is a unique morphism $F : p(X) \to L$ in $\mathbf{Law}_n(S)$ such that $\lceil F \rceil(\rho_X) = x$. In particular, we have the following bijection, natural in X and L.*

$$\lceil \mathscr{L} \rceil(1, \lceil L \rceil(X)) \;\cong\; \mathbf{Law}_n(S)(p(X), L)$$

6 RELATIVE MONADS AND THEORIES

As in the first-order setting [Linton 1969], there is a correspondence between higher-order algebraic theories and a class of monads; and correspondingly between term algebras and monad algebras. Before giving the proof, we give some intuition for how the correspondence may be understood.

Each first-order theory L corresponds to a monad T on $\mathbf{Set}^S$, which may be thought of as taking a sorted set of variables X, viewed as a context, and producing the sorted set of terms $T(X)$, in the context X, closed under the rules of the corresponding algebraic theory. The monad T is necessarily determined by its action on the finite sorted sets: formally, it is sifted-cocontinuous, which corresponds to the fact that terms may only refer to a finite number of variables.

The contexts of higher-order algebraic theories have more structure than those of first-order algebraic theories, corresponding to metavariables, metametavariables, and so on. Similarly, higher-order algebraic theories permit new operations, not just constants, to be formed by the rules of a theory. The structure of the contexts of an $(n+1)^{\text{th}}$-order algebraic theory are described by n^{th}-order algebraic theories, as discussed in Section 5.4. The monads corresponding to $(n+1)^{\text{th}}$-order algebraic theories therefore naturally take values in $\mathbf{Law}_n(S)$: we establish a correspondence between $(n+1)^{\text{th}}$-order algebraic theories and monads on the category of n^{th}-order algebraic theories. As in the first-order setting, the monads are determined by their action on the finite contexts, and as such are sifted-cocontinuous. They may be thought of similarly as closing a set of (higher-order) variables under the operations of the theory.

We obtain a monad–theory correspondence in two stages. In this section we show that higher-order algebraic theories correspond to a class of relative monads. Relative monads are similar to monads, but are defined with respect to functor $p : \mathscr{C}' \to \mathscr{C}$, typically a subcategory inclusion [Altenkirch, Chapman, and Uustalu 2010]. The restricted domain of a relative monad corresponds to the finite context limitation imposed by sifted-cocontinuity. This allows us to exchange an abstract property describing certain preservation of colimits with a more direct and conceptual description. After establishing a correspondence for relative monads, we show that these relative monads are equivalent to a class of (non-relative) monads: this takes place in Section 8 after developing prerequisite results regarding the structure of higher-order algebraic theories in Section 7.

6.1 Preliminaries

We begin with some preliminaries on relative monads that we use for the correspondence.

Definition 6.1 (Altenkirch, Chapman, and Uustalu [2010]). A *relative monad* $(T, \eta, (-)^\dagger)$ on a functor $p : \mathscr{C}' \to \mathscr{C}$ consists of an object $T(X) \in \mathscr{C}$ and morphism $\eta_X : p(X) \to T(X)$ for each $X \in \mathscr{C}'$, and a morphism $f^\dagger : T(X) \to T(Y)$ for each $f : p(X) \to T(Y)$, satisfying the following laws.

$$\eta_X{}^\dagger = \mathrm{id}_{T(X)} \qquad f^\dagger \circ \eta_X = f \qquad g^\dagger \circ f^\dagger = (g^\dagger \circ f)^\dagger$$

A *morphism* $m : (T, \eta, (-)^\dagger) \to (T', \eta', (-)^\dagger)$ of relative monads on p consists of a morphism $m_X : T(X) \to T'(X)$ for each $X \in \mathscr{C}'$, such that $m_X \circ \eta_X = \eta'_X$ and $(m_Y \circ f)^\dagger \circ m_X = m_Y \circ f^\dagger$. Relative monads on p and their morphisms form a category $\mathbf{RMnd}(p)$.

Every relative monad on $p : \mathscr{C}' \to \mathscr{C}$ induces a functor $T : \mathscr{C}' \to \mathscr{C}$ by defining $T(f) \stackrel{\text{def}}{=} (\eta \circ p(f))^\dagger$. This implies naturality of the unit η and Kleisli extension $(-)^\dagger$, as well as of any relative monad morphism. Just as with monads, relative monads have associated notions of Kleisli and Eilenberg–Moore categories. In the following, let T be a relative monad on $p : \mathscr{C}' \to \mathscr{C}$.

Definition 6.2 (Altenkirch, Chapman, and Uustalu [2010]). The *Kleisli category* $\mathbf{Kl}(T)$ of T has the same objects as $\mathscr{C}'$, and hom-sets $\mathbf{Kl}(T)(X, Y) = \mathscr{C}(p(X), T(Y))$. The identity on X is η_X, and the composition of $g \in \mathbf{Kl}(T)(Y, Z)$ and $f \in \mathbf{Kl}(T)(X, Y)$ is $g^\dagger \circ f$.

Definition 6.3 (Altenkirch, Chapman, and Uustalu [2010]). A *T-algebra* $(A, (-)^\ddagger)$ consists of an object $A \in \mathscr{C}$, and a morphism $f^\ddagger : T(X) \to A$ for each $f : p(X) \to A$, such that $f^\ddagger \circ \eta_X = f$ and $g^\ddagger \circ f^\dagger = (g^\ddagger \circ f)^\ddagger$. A *homomorphism* $h : (A, (-)^\ddagger) \to (A', (-)^\ddagger)$ of T-algebras is a morphism

$h : A \to A'$ of $\mathscr{C}$ such that $(h \circ f)^\ddagger = h \circ f^\ddagger$. These form a category T-**Alg**, which has a forgetful functor T-**Alg** $\to \mathscr{C}$.

Relative monads have a semantics functor analogous to TmAlg for theories (cf. Lemma 5.14).

Lemma 6.4. *There is a functor* Alg : $\mathbf{RMnd}(p)^{\mathrm{op}} \to \mathbf{CAT}/\mathscr{C}$ *that assigns to each relative monad* T *on* $p : \mathscr{C}' \to \mathscr{C}$ *the forgetful functor* T-**Alg** $\to \mathscr{C}$ *from its category of algebras. Moreover,* Alg *is fully faithful, and in particular reflects isomorphisms.*

6.2 Relative monads from theories

Given an $(n+1)^{\text{th}}$-order algebraic theory $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ for $n \in \mathbb{N}_\omega$, we construct a relative monad T_L on $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ (where p is as defined in Section 5.5). The assignment on objects maps $X \in \mathbb{L}_{n+1}(S)$ to the n^{th}-order algebraic theory $T_L(X) = \lfloor L /\!\!/ X \rfloor$ (using the simple slice from Definition 5.21). We write $\mathscr{T}_L(X)$ for the underlying category of $T_L(X)$; explicitly it has the same objects as $\mathbb{L}_n(S)$, and hom-sets $\mathscr{T}_L(X)(Y, Y') = \mathscr{L}(X \times Y, Y')$. The identity-on-objects functor $T_L(X)$ sends $f \in \mathbb{L}_n(S)(Y, Y')$ to $L(f) \circ \pi_Y$. The crucial property of T_L is that there are bijections

$$\mathbf{Kl}(T_L)(Y, X) \;=\; \mathbf{Law}_n(S)(p(Y), T_L(X)) \;\cong\; \mathscr{L}(X, Y) \tag{12}$$

To make T_L into a relative monad we define the unit and Kleisli extension from identities and composition in $\mathscr{L}$ using these bijections:

$$\frac{\mathrm{id}_X : X \to X \quad \text{in } \mathscr{L}}{\overline{\overline{\eta_X : p(X) \to T_L(X) \quad \text{in } \mathbf{Law}_n(S)}}} \qquad \begin{array}{ll} F : p(Y) \to T_L(X) & \text{in } \mathbf{Law}_n(S) \\ \hline\hline X \to Y & \text{in } \mathscr{L} \\ \hline F^\dagger : T_L(Y) \to T_L(X) & \text{in } \mathbf{Law}_n(S) \end{array}$$

The unit is defined by sending id_X along (12). The Kleisli extension sends F along (12) and then composes in $\mathscr{L}$. The bijections (12) form an isomorphism of categories $\mathbf{Kl}(T_L)^{\mathrm{op}} \cong \mathscr{L}$. Each map $L \to L'$ of $(n+1)^{\text{th}}$-order algebraic theories restricts to a relative monad morphism $T_L \to T_{L'}$.

Lemma 6.5. *The above defines a functor* $\mathbf{Law}_{n+1}(S) \to \mathbf{RMnd}(p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S))$ *for each* $n \in \mathbb{N}_\omega$. *For each* $(n+1)^{th}$*-order algebraic theory* $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$*, there is an isomorphism of categories* T_L-**Alg** $\cong L$-**TmAlg** *commuting with the forgetful functors into* $\mathbf{Law}_n(S)$.

6.3 Theories from relative monads

We now go in the reverse direction. As the isomorphisms $\mathbf{Kl}(T_L)^{\mathrm{op}} \cong \mathscr{L}$ in the previous section suggest, we use the opposite of the Kleisli category to do so. The construction is simple, but it turns out that we need to impose an extra condition on the relative monad to get an algebraic theory.

Suppose that $(T, \eta, (-)^\dagger)$ is a relative monad on $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$, for $n \in \mathbb{N}_\omega$. The category $\mathbf{Kl}(T)^{\mathrm{op}}$ has the same objects as $\mathbb{L}_{n+1}(S)$, and, given a morphism $f \in \mathbb{L}_{n+1}(S)(X, Y)$, we have $\eta_X \circ p(f) \in \mathbf{Law}_n(S)(p(Y), T(X)) = \mathbf{Kl}(T)^{\mathrm{op}}(X, Y)$. This defines a strict cartesian identity-on-objects functor $L_T : \mathbb{L}_{n+1}(S) \to \mathbf{Kl}(T)^{\mathrm{op}}$.

However, L_T does not in general preserve exponentials, so may not be an $(n+1)^{\text{th}}$-order algebraic theory for $n > 0$. More specifically, suppose that $n > 0$ and consider the coproduct $L' + p(Y)$ for an arbitrary $L' \in \mathbf{Law}_n(S)$ and $Y \in \mathbb{L}_n(S) \hookrightarrow \mathbb{L}_{n+1}(S)$. This coproduct is isomorphic to the simple slice $L' /\!\!/ Y$, and so has an explicit description in which hom-sets have the form $\mathscr{L}'(Y \times Z, Z')$. For T_L to be an $(n+1)^{\text{th}}$-order algebraic theory, the first and last sets in the following chain must be isomorphic.

$$\mathbf{Kl}(T)^{\mathrm{op}}(X, Z^Y) \cong \lceil T(X) \rceil (Y, Z) \cong \lceil T(X) + p(Y) \rceil (1, Z) \stackrel{?}{\cong} \lceil T(X \times Y) \rceil (1, Z) \cong \mathbf{Kl}(T)^{\mathrm{op}}(X \times Y, Z)$$

The three unmarked isomorphisms follow from the universal property of p (Lemma 5.22), the description of $L' + p(Y)$ above, and because $\lceil - \rceil$ preserves coproducts. However, the marked isomorphism does not hold in general, so, to ensure T_L is an n^{th}-order algebraic theory, we require $T(X \times Y)$ to form a coproduct $T(X) + p(Y)$. This property holds for the relative monads T_L we construct from theories L in Section 6.2, since, from the above description of the coproducts $L' + p(Y)$ as $L' /\!\!/ Y$, we have the following chain of equalities.

$$(T_L(X) /\!\!/ Y)(Z, Z') \;=\; T_L(X)(Y \times Z, Z') \;=\; \mathscr{L}(X \times Y \times Z, Z') \;=\; T_L(X \times Y)(Z, Z')$$

Lemma 6.6. *Suppose that $(T, \eta, (-)^\dagger)$ is a relative monad on $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$, where $n \in \mathbb{N}_\omega$. If $n > 0$, also assume for each $X \in \mathbb{L}_{n+1}(S)$ and $Y \in \mathbb{L}_n(S)$ that the diagram*

$$T(X) \xrightarrow{T(\pi_X)} T(X \times Y) \xleftarrow{T(\pi_Y) \circ \eta_Y} p(Y)$$

is a coproduct in $\mathbf{Law}_n(S)$. Then L_T as defined above is an $(n+1)^{th}$-order algebraic theory, and there is an isomorphism of categories T-$\mathbf{Alg} \cong L_T$-$\mathbf{TmAlg}$ commuting with the forgetful functors into $\mathbf{Law}_n(S)$. Moreover, relative monad morphisms $T \to T'$ induce morphisms $L_T \to L_{T'}$ in $\mathbf{Law}_{n+1}(S)$ functorially.

It follows from Lemma 6.5 and Lemma 6.6 that there is an isomorphism T-$\mathbf{Alg} \cong T_{L_T}$-$\mathbf{Alg}$ over $\mathbf{Law}_n(S)$, and hence, by Lemma 6.4, an isomorphism $T_{L_T} \cong T$. Given $L \in \mathbf{Law}_n(S)$, we also have $L \cong L_{T_L}$, since $\mathbf{Kl}(T_L)^{\mathrm{op}} \cong \mathscr{L}$. Hence we obtain a correspondence between relative monads and higher-order algebraic theories.

Theorem 6.7. *For $n \in \mathbb{N}_\omega$, the category $\mathbf{Law}_{n+1}(S)$ is equivalent to the full subcategory of $\mathbf{RMnd}(p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S))$ on relative monads $(T, \eta, (-)^\dagger)$ such that, if $n > 0$, then, for all $X \in \mathbb{L}_{n+1}(S)$ and $Y \in \mathbb{L}_n(S)$,*

$$T(X) \xrightarrow{T(\pi_X)} T(X \times Y) \xleftarrow{T(\pi_Y) \circ \eta_Y} p(Y)$$

is a coproduct in $\mathbf{Law}_n(S)$. Moreover, there are isomorphisms between the respective categories of algebras, commuting with the forgetful functors:

$$\begin{tikzcd} T\text{-}\mathbf{Alg} \arrow[rr, "\cong"] \arrow[dr] & & L\text{-}\mathbf{TmAlg} \arrow[dl] \\ & \mathbf{Law}_n(S) & \end{tikzcd}$$

When $n = 1$ and $S = 1$, this theorem specialises to the result that Lawvere theories are equivalent to $(\mathbf{FinSet} \hookrightarrow \mathbf{Set})$-relative monads (cf. Voevodsky [2016]).

7 LOCAL STRONG PRESENTABILITY

We will show that the categories $\mathbf{Law}_n(S)$ of n^{th}-order algebraic theories are well-structured, in that they are locally strongly presentable [Adámek and Rosický 2001; Lack and Rosický 2011]. This in turn implies several useful properties of $\mathbf{Law}_n(S)$, which in particular will be instrumental in establishing the monad correspondence in Section 8. Local strong presentability is similar to the notion of local presentability (cf. [Adámek and Rosický 1994]), except that filtered colimits are replaced by the more general notion of sifted colimits. In the following, all categories will be assumed locally small.

Definition 7.1 (Adámek and Rosický [2001]). A small category $\mathbb{I}$ is *sifted* if colimits of diagrams of shape $\mathbb{I}$ commute with finite products in $\mathbf{Set}$: explicitly if, for every finite discrete category $\mathbb{J}$ and functor $D : \mathbb{I} \times \mathbb{J} \to \mathbf{Set}$, the canonical function

$$\operatorname{colim}_{i \in \mathbb{I}} \prod_{j \in \mathbb{J}} D(i, j) \xrightarrow{[\prod_{j \in \mathbb{J}} \iota_i]_{i \in \mathbb{I}}} \prod_{j \in \mathbb{J}} \operatorname{colim}_{i \in \mathbb{I}} D(i, j)$$

is a bijection. A colimit of a diagram $D : \mathbb{I} \to \mathscr{C}$ is *sifted* when $\mathbb{I}$ is sifted.

Definition 7.2 (Adámek and Rosický [2001]). Let $\mathscr{C}$ be a category with sifted colimits. An object $X \in \mathscr{C}$ is *strongly finitely presentable* if the hom-functor $\mathscr{C}(X, -) : \mathscr{C} \to \mathbf{Set}$ is sifted-cocontinuous, i.e. preserves sifted colimits.

The strongly finitely presentable objects of **Set**, for instance, are the finite sets, whilst the strongly finitely presentable objects of $\mathbf{Set}^S$ are the indexed sets $(X_s)_{s \in S}$ such that the coproduct $\coprod_{s \in S} X_s$ is finite.

Definition 7.3 ([Adámek and Rosický 2001; Lack and Rosický 2011]). A cocomplete category $\mathscr{C}$ is *locally strongly finitely presentable* if there is a set G of strongly finitely presentable objects whose closure under sifted colimits is $\mathscr{C}$.

The strongly finitely presentable objects of $\mathscr{C}$ are given by the (essentially small) closure under retracts of the full subcategory on G. Hence, every locally strongly finitely presentable category $\mathscr{C}$ has a small full subcategory $p : \mathscr{C}_{\mathrm{sf}} \hookrightarrow \mathscr{C}$ of strongly finitely presentable objects such that every strongly finitely presentable object of $\mathscr{C}$ is isomorphic to one in $\mathscr{C}_{\mathrm{sf}}$. For our purposes, it is convenient to characterise local strong presentability directly in terms of the subcategory inclusion p. For each $X \in \mathscr{C}$, the comma category $p \downarrow X$ is sifted and the canonical morphism

$$\operatorname{colim}\,(p \downarrow X \to \mathscr{C}_{\mathrm{sf}} \xrightarrow{p} \mathscr{C}) \to X \tag{13}$$

is an isomorphism, which characterises the objects of $\mathscr{C}$ as canonical sifted colimits of objects of $\mathscr{C}_{\mathrm{sf}}$. The morphism (13) is invertible exactly when p is dense, i.e. when the nerve functor $N_p : \mathscr{C} \to [\mathscr{C}_{\mathrm{sf}}^{\mathrm{op}}, \mathbf{Set}]$, given by $N_p(X) = \mathscr{C}(p(-), X)$, is fully faithful. A category $\mathscr{C}$ is therefore locally strongly finitely presentable exactly when there exists a small category $\mathscr{C}_{\mathrm{sf}}$ and functor $p : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$ with the following property.

Definition 7.4. Let $\mathscr{C}$ be a cocomplete category and let $\mathscr{C}_{\mathrm{sf}}$ be a small category. A functor $p : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$ is *locally strongly finitely presentable* if p is fully faithful and dense, the comma category $p \downarrow X$ is sifted for every $X \in \mathscr{C}$, and every object in the image of p is strongly finitely presentable.

It is known that $\mathbf{Law}_0(S) = \mathbf{Set}^S$ is locally strongly finitely presentable. Here we show that $\mathbf{Law}_n(S)$ is furthermore locally strongly finitely presentable for $n > 0$ (which in turn implies $\mathbf{Law}_n(S)$ is locally countably presentable, but not necessarily locally finitely presentable [Adámek and Rosický 2001, Remark 4.8]). Specifically, we show that $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$, defined in Section 5.5, is locally strongly finitely presentable.

Arbitrary colimits in $\mathbf{Law}_n(S)$ are difficult to construct directly. We instead describe limits and sifted colimits in $\mathbf{Law}_n(S)$: cocompleteness then follows from the properties of p by Adámek and Rosický [2001, Remark 4.8]. We construct sifted colimits in $\mathbf{Law}_n(S)$ on each hom-set separately. Let $\mathbb{I}$ be a sifted category, and consider theories $L_i : \mathbb{L}_n(S) \to \mathscr{L}_i$ forming a diagram $D : \mathbb{I} \to \mathbf{Law}_n(S)$. The colimit of D is given by an identity-on-objects functor $L : \mathbb{L}_n(S) \to \mathscr{L}$, where $\mathscr{L}(X, Y) = \operatorname{colim}_{i \in \mathbb{I}}(\mathscr{L}_i(X, Y))$. Both composition in $\mathscr{L}$, and the action of L on morphisms $f \in \mathbb{L}_n(S)(X, Y)$, are defined using the commutativity of sifted colimits with products, as follows.

$$\circ : \mathscr{L}(Y, Z) \times \mathscr{L}(X, Y) \xrightarrow{\cong} \operatorname{colim}_{i \in \mathbb{I}}(\mathscr{L}_i(Y, Z) \times \mathscr{L}_i(X, Y)) \xrightarrow{\operatorname{colim}_{i \in \mathbb{I}}((h,g) \mapsto h \circ g)} \mathscr{L}(X, Z)$$

$$L(f) : 1 \xrightarrow{\cong} \operatorname{colim}_{i \in \mathbb{I}} 1 \xrightarrow{\operatorname{colim}_{i \in \mathbb{I}} L_i(f)} \mathscr{L}(X, Y)$$

The coprojections $\varkappa_i : L_i \to L$ are given on each hom-set by the coprojections in **Set**. Limits of small diagrams in $\mathbf{Law}_n(S)$ are described similarly, where $\mathscr{L}(X, Y) = \lim_{i \in \mathbb{I}}(\mathscr{L}_i(X, Y))$, since limits commute with products.

The requisite properties of p follow from its universal property (Lemma 5.22). In particular, for density of p it suffices to show that the functor $L \mapsto \lceil \mathscr{L} \rceil(1, \lceil L \rceil(-))$, which is naturally isomorphic to the nerve functor N_p, is fully faithful. This is the case, because given any natural transformation $\lceil \mathscr{L} \rceil(1, \lceil L \rceil(-)) \Rightarrow \lceil \mathscr{L}' \rceil(1, \lceil L' \rceil(-))$ there is a map $L \to L'$ in $\mathbf{Law}_n(S)$ given by the following composite; here we use the fact that all objects of $\mathbb{L}_n(S)$ are exponentiable in $\mathbb{L}_{n+1}(S)$.

$$\mathscr{L}(X, Y) \;\cong\; \lceil \mathscr{L} \rceil(X, Y) \;\cong\; \lceil \mathscr{L} \rceil(1, Y^X) \;\to\; \lceil \mathscr{L}' \rceil(1, Y^X) \;\cong\; \lceil \mathscr{L}' \rceil(X, Y) \;\cong\; \mathscr{L}'(X, Y)$$

Theorem 7.5. *For all $n \in \mathbb{N}_\omega$ the category $\mathbf{Law}_n(S)$ is cocomplete, and the functor $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ is locally strongly finitely presentable. Hence $\mathbf{Law}_n(S)$ is locally strongly finitely presentable.*

Completeness and cocompleteness of $\mathbf{Law}_n(S)$ signify that n^{th}-order algebraic theories may be combined, for example by taking coproducts and products. This suggests fruitful applications for algebraic effects. For instance, continuations may be presented alternately by *large algebraic theories* Hyland, Levy, Plotkin, and Power [2007], and by third-order algebraic theories (Example 3.18). However coproducts of large algebraic theories do not exist in general, while coproducts of higher-order theories always exist. This suggests that it may be more natural to combine continuations with other effects in their capacity as higher-order algebraic theories.

We note that for each n^{th}-order algebraic theory $L : \mathbb{L}_n(S) \to \mathscr{L}$, Proposition 5.15 implies that L-$\mathbf{TmAlg}$ is equivalent to the sifted cocompletion of $\mathscr{L}$ [Adámek and Rosický 2001, Corollary 2.8]. This in turn implies the following.

Theorem 7.6 (Bicompleteness of term algebras and strict models). *Let $L : \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. L-$\mathbf{TmAlg}$ is locally strongly finitely presentable, and in particular complete and cocomplete. $L/\mathbf{Law}_n(S)$ is therefore also complete and cocomplete.*

8 MONAD–THEORY CORRESPONDENCE

We now return to the task of showing that higher-order algebraic theories correspond to a class of monads. This follows from a general relationship between relative monads and monads.

Let $p : \mathscr{C}' \to \mathscr{C}$ be a functor. Each monad (T, η, μ) on $\mathscr{C}$ induces a relative monad on p, given on objects by $T \circ p : \mathscr{C}' \to \mathscr{C}$; the unit is η, and the Kleisli extension of $f : p(X) \to T(p(Y))$ is $f^\dagger = \mu_{p(Y)} \circ T(f)$. We further obtain a functor T-$\mathbf{Alg} \to (T \circ p)$-$\mathbf{Alg}$ mapping (A, a) to $(A, a \circ T(-))$, where T-$\mathbf{Alg}$ is the Eilenberg–Moore category of T.

Now suppose that $p : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$ is locally strongly finitely presentable (Definition 7.4). Then the construction of a relative monad $T \circ p$ from T forms an equivalence between relative monads on p and sifted-cocontinuous monads on $\mathscr{C}$ as follows, where a monad is *sifted-cocontinuous* if its underlying endofunctor is. The functor $(- \circ p) : [\mathscr{C}, \mathscr{C}] \to [\mathscr{C}_{\mathrm{sf}}, \mathscr{C}]$ has a left adjoint that sends each $F : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$ to its left Kan extension $\mathrm{Lan}_p F : \mathscr{C} \to \mathscr{C}$ along p. Each $\mathrm{Lan}_p F$ preserves sifted colimits, and this adjunction restricts to an equivalence $[\mathscr{C}_{\mathrm{sf}}, \mathscr{C}] \simeq [\mathscr{C}, \mathscr{C}]_{\mathrm{sf}}$, where $[\mathscr{C}, \mathscr{C}]_{\mathrm{sf}}$ is the full subcategory of $[\mathscr{C}, \mathscr{C}]$ on the sifted-cocontinuous functors. The construction of a relative monad from a monad forms a functor $(- \circ p) : \mathbf{Mnd}(\mathscr{C}) \to \mathbf{RMnd}(p)$, where $\mathbf{Mnd}(\mathscr{C})$ is the category of monads and monad morphisms, which has a left adjoint given by left Kan extension $\mathrm{Lan}_p : \mathbf{RMnd}(p) \to \mathbf{Mnd}(\mathscr{C})$ [Altenkirch, Chapman, and Uustalu 2010, Section 4.3]. Finally, when a monad T is sifted-cocontinuous, the functor T-$\mathbf{Alg} \to (T \circ p)$-$\mathbf{Alg}$ defined above is an isomorphism of categories. Together, this determines the following equivalence, where $\mathbf{Mnd}_{\mathrm{sf}}(\mathscr{C})$ is the full subcategory of $\mathbf{Mnd}(\mathscr{C})$ on the sifted-cocontinuous monads.

Theorem 8.1. *Suppose that $p : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$ is locally strongly finitely presentable. The construction above forms an adjunction (on the left), which restricts to an equivalence of categories on the sifted-cocontinuous monads (on the right).*

$$\begin{tikzcd} \mathbf{RMnd}(p) \arrow[r, shift left, "\mathrm{Lan}_p"] \arrow[r, phantom, "\perp"] & \mathbf{Mnd}(\mathscr{C}) \arrow[l, shift left, "-\circ p"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{RMnd}(p) \arrow[r, shift left, "\mathrm{Lan}_p"] \arrow[r, phantom, "\simeq"] & \mathbf{Mnd}_{\mathrm{sf}}(\mathscr{C}) \arrow[l, shift left, "-\circ p"] \end{tikzcd}$$

Moreover, there are isomorphisms between the corresponding categories of algebras, and these commute with the forgetful functors, as below, for all $T \in \mathbf{Mnd}_{\mathrm{sf}}(\mathscr{C})$ and $T' \in \mathbf{RMnd}(p)$.

$$\begin{tikzcd} T\text{-}\mathbf{Alg} \arrow[rr, "\cong"] \arrow[dr] & & (T \circ p)\text{-}\mathbf{Alg} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd} \qquad \begin{tikzcd} (\mathrm{Lan}_p T')\text{-}\mathbf{Alg} \arrow[rr, "\cong"] \arrow[dr] & & T'\text{-}\mathbf{Alg} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

For $(n+1)^{\text{th}}$-order algebraic theories, $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ is locally strongly finitely presentable by Theorem 7.5. The equivalence between $(n+1)^{\text{th}}$-order algebraic theories and a class of relative monads on p given by Theorem 6.7 extends to a class of monads by Theorem 8.1, and we obtain the final monad–theory correspondence.

Theorem 8.2. *For $n \in \mathbb{N}_\omega$, the following are equivalent.*

1. *The category $\mathbf{Law}_{n+1}(S)$ of $(n+1)^{th}$-order algebraic theories.*
2. *The full subcategory of $\mathbf{RMnd}(p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S))$ on relative monads $(T, \eta, (-)^\dagger)$ such that, if $n > 0$, then, for all $X \in \mathbb{L}_{n+1}(S)$ and $Y \in \mathbb{L}_n(S)$,*
$$T(X) \xrightarrow{T(\pi_X)} T(X \times Y) \xleftarrow{T(\pi_Y) \circ \eta_Y} p(Y)$$
is a coproduct in $\mathbf{Law}_n(S)$.
3. *The full subcategory of $\mathbf{Mnd}_{\mathrm{sf}}(\mathbf{Law}_n(S))$ on monads (T, η, μ) such that, if $n > 0$, then, for all $L \in \mathbf{Law}_n(S)$ and $Y \in \mathbb{L}_n(S)$,*
$$T(L) \xrightarrow{T(\iota_L)} T(L + p(Y)) \xleftarrow{T(\iota_{p(Y)}) \circ \eta_{p(Y)}} p(Y)$$
is a coproduct in $\mathbf{Law}_n(S)$.

Moreover, if an $(n+1)^{th}$-order algebraic theory L, relative monad T, and monad $\hat{T}$ are related by these equivalences, then there are isomorphisms between the respective categories of categories of algebras commuting with the forgetful functors:

$$\begin{tikzcd} \hat{T}\text{-}\mathbf{Alg} \arrow[r, "\cong"] \arrow[dr] & T\text{-}\mathbf{Alg} \arrow[r, "\cong"] \arrow[d] & L\text{-}\mathbf{TmAlg} \arrow[dl] \\ & \mathbf{Law}_n(S) & \end{tikzcd}$$

Example 8.3. Consider the $\{\mathsf{U}\}$-sorted second-order algebraic theory of the unityped λ-calculus (Example 3.11). This theory induces a monad on $\mathbf{Law}_1(\{\mathsf{U}\})$, which sends each first-order algebraic theory L to the first-order algebraic theory $L' : \mathbb{L}_1(\{\mathsf{U}\}) \to \mathscr{L}'$ underlying the free term algebra for the unityped λ-calculus on L. In other words, L' is initial amongst first-order algebraic theories equipped with a map $L \to L'$ interpreting L in L', and two families of functions,

$$[\![\mathsf{abs}]\!]_X : \mathscr{L}'(X \times \mathsf{U}, \mathsf{U}) \to \mathscr{L}'(X, \mathsf{U}) \qquad [\![\mathsf{app}]\!]_X : \mathscr{L}'(X, \mathsf{U}) \times \mathscr{L}'(X, \mathsf{U}) \to \mathscr{L}'(X, \mathsf{U})$$

as in Example 5.13. In the terminology of Hyland [2017], L' is the free λ-theory (with η) on L.

For instance, when L is the first-order theory of semigroups (which has a single associative binary operator), the map $L \to L'$ amounts to a natural family of associative functions,

$$[\![\mathsf{choose}]\!]_X : \mathscr{L}'(X, \mathsf{U}) \times \mathscr{L}'(X, \mathsf{U}) \to \mathscr{L}'(X, \mathsf{U})$$

which may be thought of as a binary nondeterministic choice operation. The set $\mathscr{L}'(\mathsf{U}^n, \mathsf{U})$ is in bijection with the set of open terms of the unityped λ-calculus with a binary nondeterministic choice operator choose, with at most n free variables, up to $\beta\eta$-equality and associativity of choose.

8.1 Explicit formulae for monads induced by theories

It is well-known that, for a first-order algebraic theory $L : \mathbb{L}_1(S) \to \mathscr{L}$, the corresponding sifted-cocontinuous monad $\hat{T}_L$ on $\mathbf{Set}^S$ may be defined as a coend [Hyland and Power 2007, Proposition 4.1]. Using the universal property of p (Lemma 5.22), we may present it slightly differently:

$$\hat{T}_L(L')(\mathsf{B}) \;\cong\; \int^{\Gamma \in \mathbb{L}_1(S)} \mathscr{L}(\Gamma, \mathsf{B}) \times \lceil \mathscr{L}' \rceil(1, \Gamma)$$

An $(n+1)^{\text{th}}$-order algebraic theory $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$, for $n > 0$, induces a monad $\hat{T}_L$ on $\mathbf{Law}_n(S)$, given by the left Kan extension of T_L along p. Since left Kan extensions along p are sifted colimits, which in $\mathbf{Law}_n(S)$ are computed componentwise, there is an analogous coend formula for $n > 0$:

$$\hat{T}_L(L')(X, Y) \;\cong\; \int^{\Gamma \in \mathbb{L}_{n+1}(S)} \mathscr{L}(\Gamma, Y^X) \times \lceil \mathscr{L}' \rceil(1, \Gamma)$$

There is an alternate characterisation, making use of the coreflections of Section 5.2, which is particularly elegant. Since the coreflections are obscured when one considers only first-order theories, this description is new even in the classical setting.

$$\hat{T}_L \;\cong\; \lfloor L + \lceil - \rceil \rfloor$$

In this form, it is natural to think of higher-order algebraic theories as certain monad transformers [Liang, Hudak, and Jones 1995]. For instance, for a second-order algebraic theory L, the underlying endofunctor of induced monad on $\mathbf{Law}_1(S)$ itself induces a functor that, informally, takes a monad and freely adds the structure of L by taking its coproduct with L:

$$\mathbf{Mnd}_{\mathrm{sf}}(\mathbf{Set}^S) \xrightarrow{\simeq} \mathbf{Law}_1(S) \xrightarrow{\lfloor L + \lceil - \rceil \rfloor} \mathbf{Law}_1(S) \xrightarrow{\simeq} \mathbf{Mnd}_{\mathrm{sf}}(\mathbf{Set}^S)$$

8.2 Related work

General monad–theory correspondences. Though there are several general correspondences between notions of algebraic theory and classes of monads in the literature, none captures the correspondence here. On one hand, those of Lack and Rosický [2011]; Lucyshyn-Wright [2016]; Power [1999] are insufficiently general: our setting requires the generality of Berger, Melliès, and Weber [2012]; Bourke and Garner [2019], for which $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ exhibits $\mathbb{L}_{n+1}(S)^{\mathrm{op}}$ as a small full dense subcategory of the locally presentable $\mathbf{Law}_n(S)$. However, the correspondence they give specialises in our setting to one involving the whole of $\mathbf{Mnd}_{\mathrm{sf}}(\mathbf{Law}_n(S))$ on the monad side, without the coproduct condition of Theorem 8.2: our correspondence is therefore a restriction, which is not recovered in their framework.

Monads for binding signatures. There are several existing approaches to generating monads from signatures of variable-binding operators, e.g. Ahrens, Hirschowitz, Lafont, and Maggesi [2019]; Matthes and Uustalu [2004]. The generated monads in these approaches are analogous to those constructed in our framework by $\lfloor - \rfloor : \mathbf{Law}_2(S) \to \mathbf{Law}_1(S) \simeq \mathbf{Mnd}_{\mathrm{sf}}(\mathbf{Set}^S)$, rather than to the higher-order monads arising from our monad–theory correspondence.

9 ZEROTH-ORDER ALGEBRAIC THEORIES

For $n > 0$, the category of $(n+1)^{\text{th}}$-order algebraic theories is formed by taking a class of monads on the category of n^{th}-order algebraic theories, while the category of first-order algebraic theories is formed by taking a class of monads on $\mathbf{Set}^S$. Accordingly, it is natural to ask whether the category of S-indexed sets itself may naturally be considered a "category of 0^{th}-order algebraic theories" in some sense, which would eliminate the seemingly arbitrary base case for the induction (rather than declaring $\mathbf{Law}_0(S) = \mathbf{Set}^S$, as we have done till now). This is the case: 0^{th}-order algebraic theories correspond to *theories of constants*, in which each term resides in an empty context. In what follows, we redefine $\mathbf{Law}_0(S)$ and show it to be isomorphic to $\mathbf{Set}^S$. Our earlier identification is thus harmless.

Definition 9.1. The category $\mathbb{L}_0(S)$ is the *free nullary completion* of the set S. Concretely, S is the full supercategory of the discrete category S with object-set $S + 1$, such that there is a unique morphism from every object in S to 1, and every morphism from 1 is the identity.

Definition 9.2. An *S-sorted 0^{th}-order algebraic theory* is a category $\mathscr{L}$ with a terminal object, equipped with a strict terminal-object-preserving identity-on-objects functor $L : \mathbb{L}_0(S) \to \mathscr{L}$, such that every morphism is *constant*, i.e. factors through the terminal object. A *map* of S-sorted 0^{th}-order algebraic theories from $\mathscr{L}$ to $\mathscr{L}'$ is a functor commuting with L and L', which necessarily strictly preserves the terminal object. S-sorted 0^{th}-order algebraic theories and their maps form a category $\mathbf{Law}_0(S)$.

We note that 0^{th}-order algebraic theories may be more naturally described as generalised multicategories [Leinster 2004] for the terminal monad on **Set**, with underlying object-set S.

LEMMA 9.3. $\mathbf{Law}_0(S) \cong \mathbf{Set}^S$.

Though this characterisation may seem inconsequential, it is actually helpful in understanding the syntactic behaviour of the process of taking sifted-cocontinuous monads with the coproduct condition of Theorem 8.2. The $\mathbf{Mnd}_{\mathrm{sf}}(-)$ construction may be thought of as taking a class of theories and adding a level of parameterisation by a cartesian context. To begin, one has theories of constants, whose terms reside in empty contexts; taking sifted-cocontinuous monads results in first-order algebraic theories, whose terms are parameterised by ordinary contexts; taking sifted-cocontinuous monads a second time results in second-order algebraic theories, which have metavariables in addition to ordinary variables; each subsequent iteration adds a further level of parameterisation. This observation has immediate implications for higher-order theories with different context structure: for instance, one might suppose that higher-order linear binding structure might be constructed analogously by analytic monads [Joyal 1986]; we leave this for future work.

10 CONCLUSION

Higher-order algebraic theories generalise the classical notion of algebraic theory to higher-order multisorted structure, commonly found in mathematics and particularly in the theory of programming languages. Our results provide part of a systematic treatment of higher-order theories analogous to the treatment that first-order theories have received. There are three equivalent, but complementary, perspectives: those of presentations (Section 3), theories (Section 5), and monads (Section 8). Higher-order algebraic theories are well-structured (Section 7), as are their categories of term algebras and models (Theorem 7.6): for instance, we may take products and coproducts of arbitrary higher-order algebraic theories, notably including those for control operators, which have previously proven difficult to handle. The higher-order perspective also sheds new light on the classical, first-order setting, elucidating the relationship between models and term algebras (Section 5.4), and giving an elegant description of the monad induced by an algebraic theory (Section 8.1). We hope that this work may serve as a basis for the ongoing study of higher-order structure: some suggested applications have appeared throughout, but we note also the possibility of carrying out similar developments for other notions of algebraic theory, such as enriched algebraic theories [Power 1999], algebraic 2-theories [Yanofsky 2000] and indexed algebraic theories [Power 2011].

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A PROOFS & ADDITIONAL DEFINITIONS

A.1 Introduction

A.1.3 Preliminaries.
We denote by **Cat** the large category of small categories and functors, and by **CAT** the very large category of locally small categories and functors.

Definition A.1. Given two categories $\mathscr{C}$ and $\mathscr{D}$ whose object sets are equal, an *identity-on-objects functor* $F : \mathscr{C} \to \mathscr{D}$ is one whose object function is the identity.

Definition A.2. A category is *cartesian* if it has specified finite products. A functor $F : \mathscr{C} \to \mathscr{D}$ between cartesian categories is *cartesian* if it preserves finite products. Explicitly, this means that

- the canonical morphism $\langle\rangle : F(1_{\mathscr{C}}) \to 1_{\mathscr{D}}$ is an isomorphism;
- for all objects $X, Y \in \mathscr{C}$, the canonical morphism $\langle F(\pi_1), F(\pi_2)\rangle : F(X \times Y) \to F(X) \times F(Y)$ is an isomorphism.

A cartesian functor is *strict* if the canonical isomorphisms are identities. Cartesian categories, cartesian functors, and natural transformations form a 2-category **Cart**.

Definition A.3. A category is *cartesian-closed* if it is cartesian and has specified exponentials. A functor $F : \mathscr{C} \to \mathscr{D}$ between cartesian-closed categories is *cartesian-closed* if it is cartesian and preserves exponentials. Explicitly, this means that

- for all objects $X, Y \in \mathscr{C}$, the canonical morphism $\lambda(F(\mathrm{ev}_{\mathscr{C}}) \circ \langle F(\pi_1), F(\pi_2)\rangle^{-1}) : F(Y^X) \to F(Y)^{F(X)}$ is an isomorphism.

A cartesian-closed functor is *strict* if it is strict as a cartesian functor, and the canonical isomorphisms are identities. Cartesian-closed categories, cartesian-closed functors and natural transformations form a 2-category **CCC**.

A.3 Presentations of higher-order algebraic theories

A.3.1 The order-limited λ-calculus.
The $(n+1)^{\text{th}}$-order simply-typed λ-calculus is presented in Figure A.1.

Proposition A.4. *Let $n \in \mathbb{N}_\omega$. A product of objects $X_1 \times \cdots \times X_m$ in a cartesian category is n-tetrable iff X_i is n-tetrable, for each $m \in \mathbb{N}$ and $1 \le i \le m$.*

Proof. We proceed by induction. When $n = k+1$, we assume that $X_1 \times \cdots \times X_m$ is k-tetrable iff each X_i is k-tetrable. Then we have $X_1 \times \cdots \times X_m \upuparrows (k+1) = (X_1 \times \cdots \times X_m)^{X_1 \times \cdots \times X_m \upuparrows k} \cong (X_1^{X_1 \times \cdots \times X_m \upuparrows k}) \times \cdots \times (X_m^{X_1 \times \cdots \times X_m \upuparrows k})$, by the universal property of the product. By currying, a product is exponentiable iff each component is exponentiable, and so by the inductive hypothesis $X_1 \times \cdots \times X_m$ is $(k+1)$-tetrable if each X_i is. Finally, $X_1 \times \cdots \times X_m$ is trivially 0-tetrable, and so we are done. □

Lemma A.5. *$\Lambda_n(S)$ is cartesian, and contains S as an n-tetrable subcategory.*

Proof. The unit type 1 is terminal, as for each context $x : X$, there is a term $x : X \vdash \langle\rangle : 1$, which is unique by the η-law. The cartesian product of any pair of types X and Y is given by the product type $X \times Y$, with projections given by variable projections for X and Y, and mediating morphism given by pairing; satisfaction of the universal property follows from the β- and η-laws. S is a subcategory by construction. Every $B \in S$ in n-tetrable: tetrations of B are given by function types, with the evaluation maps given by application, and the exponential transpose given by λ-abstraction; satisfaction of the universal property follows from the β- and η-laws. □

$$\mathrm{ord}(B) \stackrel{\text{def}}{=} 1 \qquad (B \in S)$$

$$\mathrm{ord}(1) \stackrel{\text{def}}{=} 0$$

$$\mathrm{ord}(X \times Y) \stackrel{\text{def}}{=} \max(\mathrm{ord}(X), \mathrm{ord}(Y))$$

$$\mathrm{ord}(X \to Y) \stackrel{\text{def}}{=} \max(\mathrm{ord}(X) + 1, \mathrm{ord}(Y))$$

$$\frac{}{\cdot \ \mathsf{ctx}}\ \textsc{empty} \qquad \frac{\Gamma\ \mathsf{ctx} \qquad X\ \mathsf{type}}{\Gamma, x : X\ \mathsf{ctx}}\ \textsc{ext} \qquad \frac{}{\Gamma, x : X, \Delta \vdash x : X}\ \textsc{var}$$

$$\frac{}{B\ \mathsf{type}}\ (B \in S)\ \textsc{base}$$

$$\frac{}{1\ \mathsf{type}}\ 1\text{-}\textsc{form} \qquad \frac{X\ \mathsf{type} \qquad Y\ \mathsf{type}}{X \times Y\ \mathsf{type}}\ \times\text{-}\textsc{form} \qquad \frac{X\ \mathsf{type} \qquad \mathrm{ord}(X) < n \qquad Y\ \mathsf{type}}{X \to Y\ \mathsf{type}}\ \to\text{-}\textsc{form}$$

$$\frac{}{\Gamma \vdash \langle\rangle : 1}\ 1\text{-}\textsc{intro} \qquad \frac{\Gamma \vdash u : 1}{\Gamma \vdash u \equiv \langle\rangle : 1}\ 1\text{-}\eta$$

$$\frac{\Gamma \vdash a : X \qquad \Gamma \vdash b : Y}{\Gamma \vdash \langle a, b\rangle : X \times Y}\ \times\text{-}\textsc{intro} \qquad \frac{\Gamma \vdash p : X \times Y}{\Gamma \vdash \pi_1(p) : X}\ \times\text{-}\textsc{elim}_1 \qquad \frac{\Gamma \vdash p : X \times Y}{\Gamma \vdash \pi_2(p) : Y}\ \times\text{-}\textsc{elim}_2$$

$$\frac{\Gamma \vdash a : X \qquad \Gamma \vdash b : Y}{\Gamma \vdash \pi_1\langle a, b\rangle \equiv a : X}\ \times\text{-}\beta_1 \qquad \frac{\Gamma \vdash a : X \qquad \Gamma \vdash b : Y}{\Gamma \vdash \pi_2\langle a, b\rangle \equiv b : Y}\ \times\text{-}\beta_2$$

$$\frac{\Gamma \vdash p : X \times Y}{\Gamma \vdash \langle \pi_1 p, \pi_2 p\rangle \equiv p : X \times Y}\ \times\text{-}\eta$$

$$\frac{\Gamma, x : X \vdash t : Y \qquad \mathrm{ord}(X) < n}{\Gamma \vdash \lambda(x : X.\, t) : X \to Y}\ \to\text{-}\textsc{intro} \qquad \frac{\Gamma \vdash f : X \to Y \qquad \Gamma \vdash a : X}{\Gamma \vdash f\ a : Y}\ \to\text{-}\textsc{elim}$$

$$\frac{\Gamma, x : X \vdash t : Y \qquad \Gamma \vdash a : X}{\Gamma \vdash (\lambda(x : X.\, t))\ a \equiv t[{}^{a}/_{x}] : Y}\ \to\text{-}\beta \qquad \frac{\Gamma \vdash f : X \to Y}{\Gamma \vdash \lambda(x : X.\, f\ x) \equiv f : Y}\ \to\text{-}\eta$$

$t[{}^{a}/_{x}]$ denotes the substitution of the term a for the free variable x in the term t.
Context extension is defined inductively by repeated variable extension.

Fig. A.1. The $(n+1)^{\text{th}}$-order simply-typed λ-calculus on S.

Theorem 3.6. *Let $n \in \mathbb{N}_\omega$. $\Lambda_{n+1}(S)$ is the 2-initial cartesian category containing S as an n-tetrable subcategory. This exhibits $\Lambda_{n+1}(S)$ as the free cartesian category with an n-tetrable subcategory S.*

Proof. First, note that $\Lambda_n(S)$ is a cartesian category containing S as an n-tetrable subcategory by Lemma A.5. Let $\iota : S \hookrightarrow \mathbb{C}$ be a cartesian category containing S as an n-tetrable subcategory. We define a cartesian functor $F : \Lambda_n(S) \to \mathbb{C}$ preserving n-exponentiable objects in S by induction on the rules of the n^{th}-order simply-typed λ-calculus.

$$\begin{array}{ccc} F(1) = 1 & F(\langle\rangle) = \langle\rangle & \\ F(X \times Y) = F(X) \times F(Y) & F(\langle x, y\rangle) = \langle F(x), F(y)\rangle & F(\pi_i(p)) = \pi_i F(p) \\ F(X \to Y) = F(Y)^{F(X)} & F(\lambda z.t) = \lambda(F(t)) & F(f\ x) = \mathrm{ev}(F(f), F(x)) \\ F(\mathrm{B}) = \iota(B) & & \end{array}$$

Checking this is functorial and structure-preserving is routine. The definition of F was entirely determined, up to isomorphism, by the requirement that F be structure-preserving. 2-initiality follows immediately. □

A.4 Free cartesian-closed categories

A.4.1 Cartesian-closed categories of trees.

Proposition A.6. $\mathrm{Tree}_{(-)}$ *is canonically a* $(\mathbb{N}, +, 0)$*-graded monad.*

Proof. The unit $\mathrm{Id} \Rightarrow \mathrm{Tree}_0$ is given by the right coprojection; the multiplication $\mathrm{Tree}_m \circ \mathrm{Tree}_n \Rightarrow \mathrm{Tree}_{m+n}$ by tree grafting: the maximum number of left-steps is achieved by grafting the two maximal trees. Monadicity follows from that of Tree. □

Theorem 4.4. $\mathbb{L}_{n+1}(S)$ *is the initial strict cartesian category containing* S *as a strictly* n*-tetrable subcategory, for* $n \in \mathbb{N}_\omega$*.*

Proof. Each element $B \in S$ forms a singleton tree in a singleton list. Cartesian products are given in $\mathbb{L}_n(S)$ by the empty list and list concatenation, which is strictly associative and unital. Exponentials are given by appropriately currying and distributing trees of lists according to the universal properties. Currying is strict by construction. The statement then follows by the same reasoning as Theorem 3.6, except that the unique functor from $\mathbb{L}_n(S)$ is determined uniquely, since the structure is entirely strict. □

Proposition A.7. *Let* $0 < m \leq n \in \mathbb{N}_\omega$*.* $\mathbb{L}_m(S)$ *is a strictly full subcategory of* $\mathbb{L}_n(S)$*.*

Proof. By definition, $\mathrm{Col}_m(S) \subset \mathrm{Col}_n(S)$ and for any $\Gamma, \Delta \in \mathrm{Col}_m(S)$ the hom-set $\mathbb{L}_m(S)(\Gamma, \Delta) = \mathbb{L}_n(S)(\Gamma, \Delta)$. □

Proposition 4.5. *The* $(n+1)^{th}$*-order simply-typed* λ*-calculus on* S *is a (faithful) conservative extension of the* n^{th}*-order simply-typed* λ*-calculus on* S*, for* $n \in \mathbb{N}_\omega$*.*

Proof. The inclusion $\mathbb{L}_n(S) \hookrightarrow \mathbb{L}_{n+1}(S)$ is full (and faithful). □

A.5 Higher-order algebraic theories

A.5.1 Equivalence with presentations.

Lemma A.8. $\Lambda_{(-)} : \mathbf{Pre}_n(S) \to \mathbf{Law}_n(S)$ *is left adjoint to* $\Pi_{(-)}$*.*

Proof. We define a functor $\Pi_{(-)} : \mathbf{Law}_n(S) \to \mathbf{Pre}_n(S)$ sending each n^{th}-order algebraic theory $L : \mathbb{L}_n(S) \to \mathscr{L}$ to the presentation $(\coprod_{(\Gamma,A)\in\Lambda_n(S)\times S} \mathscr{L}(\Gamma, A), \pi, E)$, where E is given by identifying the formal projections and compositions with the variable projections and substitutions, and each map $F : \mathscr{L} \to \mathscr{L}'$ to the transliteration given by the function $(\Gamma, A, t) \mapsto (\Gamma, A, F(t))$. This preserves the equations in Π_L by functoriality.

Let $\Sigma \in \mathbf{Pre}_n(S)$ and $L \in \mathbf{Law}_n(S)$. A map $F : \Lambda_\Sigma \to L$ is specified entirely by the action on the operators of Σ, as the action on the derived terms is forced by functoriality of F; this may be seen to be exactly the data of a transliteration $f : \Sigma \to \Pi_L$. Thus $\mathbf{Law}_n(S)(\Lambda_\Sigma, L) \cong \mathbf{Pre}_n(S)(\Sigma, \Pi_L)$. □

Lemma A.9. $\mathbf{Pres}_n(S) \cong \mathbf{Kl}(\Pi_{(-)} \circ \Lambda_{(-)})$.

Proof. Follows directly from the definition of $\mathbf{Pres}_n(S)$. □

Theorem 5.2. $\mathbf{Law}_n(S) \simeq \mathbf{Pres}_n(S)$.

Proof. $\Lambda_{(-)}$ is essentially surjective: indeed, for every $L \in \mathbf{Law}_n(S)$, we have $\Lambda_{\Pi_L} \cong L$. Therefore $\mathbf{Law}_n(S) \simeq \mathbf{Kl}(\Pi_{(-)} \circ \Lambda_{(-)})$, which is equivalent to $\mathbf{Pres}_n(S)$ by Lemma A.9. □

Lemma A.10. *$\mathbf{Law}_n(S)$ is a reflective subcategory of $\mathbf{Pre}_n(S)$.*

Proof. $\Pi_{(-)}$ is fully faithful and so the result follows from Lemma A.8. □

Proposition A.11. $\mathbf{Law}_n(S) \simeq (\Pi_{(-)} \circ \Lambda_{(-)})\text{-}\mathbf{Alg}$.

Proof. Every reflective subcategory is equivalent to the Eilenberg–Moore category of its associated idempotent monad [Borceux 1994b, Corollary 4.2.4] and so the result follows from Lemma A.10. □

A.5.2 Coreflections between categories of theories.

Theorem 5.3. *For each $n \in \mathbb{N}_\omega$, the constructions above form an adjunction,*

$$\begin{tikzcd} \mathbf{Law}_n(S) \arrow[r, shift left=1.5, "\lceil - \rceil"] \arrow[r, phantom, "\perp"] & \mathbf{Law}_{n+1}(S) \arrow[l, shift left=1.5, "\lfloor - \rfloor"] \end{tikzcd}$$

with $\lceil - \rceil$ fully faithful. Hence $\mathbf{Law}_n(S)$ is a coreflective subcategory of $\mathbf{Law}_{n+1}(S)$.

Proof. We define a functor $\lceil - \rceil : \mathbf{Law}_n(S) \to \mathbf{Law}_{n+1}(S)$ sending an n^{th}-order algebraic theory $\mathscr{L}$ to the $(n+1)^{\text{th}}$-order algebraic theory given by the trivial embedding of $\mathrm{Pres}(\mathscr{L})$ as a $(n+1)^{\text{th}}$-order presentation. This is fully faithful, as the possible structure of a translation is entirely determined by the operators of the domain and codomain. We define a functor $\lfloor - \rfloor : \mathbf{Law}_{n+1}(S) \to \mathbf{Law}_n(S)$ sending an $(n+1)^{\text{th}}$-order algebraic theory $\mathscr{L}'$ to its full subcategory on $\mathbb{L}_n(S)$.

These functors form an adjunction $\mathbf{Law}_n(S) : \lceil - \rceil \dashv \lfloor - \rfloor : \mathbf{Law}_{n+1}(S)$. For consider an n^{th}-order algebraic theory $\mathscr{L}$ and $(n+1)^{\text{th}}$-order algebraic theory $\mathscr{L}'$: a map of $(n+1)^{\text{th}}$-order algebraic theories $F : \lceil \mathscr{L} \rceil \to \mathscr{L}'$ is nontrivial only on objects of $\mathbb{L}_n(S)$, as each map is identity-on-objects; likewise, a map of n^{th}-order algebraic theories $G : \mathscr{L} \to \lfloor \mathscr{L}' \rfloor$ is entirely determined on objects of $\mathbb{L}_n(S)$, essentially by definition. Thus $\mathbf{Law}_{n+1}(S)(\lceil \mathscr{L} \rceil, \mathscr{L}') \cong \mathbf{Law}_n(S)(\mathscr{L}, \lfloor \mathscr{L}' \rfloor)$. □

The following property of $\lceil - \rceil$ is needed later.

Lemma A.12. *The functor $L \mapsto \lceil \mathscr{L} \rceil(1, \lceil L \rceil(-)) : \mathbf{Law}_n(S) \to [\mathbb{L}_{n+1}(S), \mathbf{Set}]$ is fully faithful, for all $n \in \mathbb{N}_\omega$.*

Proof. We first prove this for the case $n > 0$. Given a natural transformation $\alpha : \lceil \mathscr{L} \rceil(1, \lceil L \rceil(-)) \Rightarrow \lceil \mathscr{L}' \rceil(1, \lceil L' \rceil(-))$, we define an identity-on-objects functor

$F_\alpha : \lfloor\lceil \mathscr{L} \rceil\rfloor \to \lfloor\lceil \mathscr{L}' \rceil\rfloor$ on morphisms as:

$$\begin{aligned}\lfloor\lceil \mathscr{L} \rceil\rfloor(X,Y) &= \lceil \mathscr{L} \rceil(X,Y) \\ &\overset{\lambda}{\cong} \lceil \mathscr{L} \rceil(1, Y^X) \\ &= \lceil \mathscr{L} \rceil(1, \lceil L \rceil(Y^X)) \\ &\xrightarrow{\alpha} \lceil \mathscr{L}' \rceil(1, \lceil L' \rceil(Y^X)) \\ &= \lceil \mathscr{L}' \rceil(1, Y^X) \\ &\overset{\lambda^{-1}}{\cong} \lceil \mathscr{L}' \rceil(X,Y) \\ &= \lfloor\lceil \mathscr{L}' \rceil\rfloor(X,Y)\end{aligned}$$

Preservation of identities and composition follows by naturality of α. Furthermore, naturality of α implies the commutativity of the following diagram, so terminality of 1 implies $\alpha_Z(\lceil L(z) \rceil) = \lceil L'(z) \rceil$, for every $z \in \mathbb{L}_{n+1}(1, Z)$.

$$\begin{tikzcd} \lceil \mathscr{L} \rceil(1,1) \arrow[r, "\lceil L(z) \rceil"] \arrow[d, "\alpha_1"'] & \lceil \mathscr{L}' \rceil(1,Z) \arrow[d, "\alpha_Z"] \\ \lceil \mathscr{L}' \rceil(1,1) \arrow[r, "\lceil L'(z) \rceil"'] & \lceil \mathscr{L}' \rceil(1,Z) \end{tikzcd}$$

This implies that $\lfloor\lceil L' \rceil\rfloor = F_\alpha \circ \lfloor\lceil L \rceil\rfloor$, hence F_α is a morphism $\lfloor\lceil L \rceil\rfloor \to \lfloor\lceil L' \rceil\rfloor$ in $\mathbf{Law}_n(S)$, and so $\eta^{-1} \circ F_\alpha \circ \eta$ is a morphism $L \to L'$ in $\mathbf{Law}_n(S)$, where η is the unit of the coreflection.

Now consider a morphism $G : L \to L'$ in $\mathbf{Law}_n(S)$. The functor maps G to the natural transformation α defined as the hom-function of $\lceil G \rceil$. Since $\lceil G \rceil$ preserves exponentials, we have $F_\alpha = \lfloor\lceil G \rceil\rfloor$, which implies $\eta^{-1} \circ F_\alpha \circ \eta = G$.

To show that composing the functions between the hom-sets in the other direction also gives the identity, consider the following diagram, for $X \in \mathbb{L}_{n+1}(S)$ and $Z \in \mathbb{L}_n(S)$.

$$\begin{tikzcd} \lceil \mathscr{L} \rceil(Z,X) \arrow[r, "\lambda"] \arrow[d, "\lceil \eta \rceil"'] & \lceil \mathscr{L} \rceil(1,X^Z) \arrow[r, "\alpha"] & \lceil \mathscr{L}' \rceil(1,X^Z) \arrow[r, "\lambda^{-1}"] & \lceil \mathscr{L}' \rceil(Z,X) \arrow[d, "\lceil \eta \rceil"] \\ \lceil\lfloor\lceil \mathscr{L} \rceil\rfloor\rceil(Z,X) \arrow[rrr, "\lceil F_\alpha \rceil"'] & & & \lceil\lfloor\lceil \mathscr{L} \rceil\rfloor\rceil(Z,X) \end{tikzcd}$$

Commutativity of the diagram will then imply the lemma statement when $Z = 1$. The diagram can be shown to commute by induction on X: if $X \in \mathbb{L}_n(S)$, commutativity follows from the definition of F_α and invertibility of the unit η; otherwise, we use naturality of α and commutativity of $\lceil \eta \rceil$ with products and exponentials.

For $n = 0$, the proof is similar, except that we construct a morphism $X \to X'$ in $\mathbf{Set}^S$:

$$X_s \cong \lceil X \rceil(1,s) \xrightarrow{\alpha} \lceil X' \rceil(1,s) \cong X'_s \qquad \square$$

Corollary A.13. *Let $0 < m \leq n \in \mathbb{N}_\omega$. $\mathbf{Law}_m(S)$ is a coreflective subcategory of $\mathbf{Law}_n(S)$.*

Proof. Proof proceeds as in Theorem 5.3 using Proposition A.7. □

Corollary A.14. *Let $D = \lceil - \rceil \circ \lfloor - \rfloor$ be the comonad induced by the adjunction*

$$\begin{tikzcd} \mathbf{Law}_n(S) \arrow[r, shift left, "\lceil - \rceil"] \arrow[r, phantom, "\perp"] & \mathbf{Law}_{n+1}(S) \arrow[l, shift left, "\lfloor - \rfloor"] \end{tikzcd}$$

There is an equivalence $\mathbf{Law}_n(S) \simeq D$-$\mathbf{Coalg}$.

Proof. Every coreflective subcategory is equivalent to the Eilenberg–Moore category of its associated idempotent comonad [Borceux 1994b, Corollary 4.2.4] and so the result follows from Theorem 5.3. □

A.5.3 Models and strict models.

Proposition 5.7. *Let $L : \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory, and let $U : \mathbf{Law}_n(S) \to \mathbf{Cart}$ be the functor forgetting the generating sorts and specified structure. The coslice category $L/\mathbf{Law}_n(S)$ is a non-full subcategory of $\int \mathbf{Mod}(L, U(-))$.*

Proof. Explicitly, the category $\int \mathbf{Mod}(L, U(-))$ has

- objects, pairs $(L' \in \mathbf{Law}_n(S), M : \mathscr{L} \to U(L'))$;
- morphisms, pairs $(F : L' \to L'', \mu : \mathscr{L} \xrightarrow{M} U(L') \xrightarrow{U(F)} U(L'') \Rightarrow M')$.

It is evident that the coslice category $L/\mathbf{Law}_n(S)$ is contained inside this one, precisely as that for which $M \circ L = L'$ and μ is the identity. □

Proposition 5.8. *$\int \mathbf{Mod}(L, -)$ is equivalent to the subcategory of the lax coslice $\mathscr{L}//\mathbf{Cart}$ for which the coslices preserve exponentials.*

Proof. Explicitly, the category $\int \mathbf{Mod}(L, -)$ has

- objects, pairs $(\mathscr{C} \in \mathbf{Cart}, M : \mathscr{L} \to \mathscr{C})$;
- morphisms, pairs $(F : \mathscr{C} \to \mathscr{C}', \mu : \mathscr{L} \xrightarrow{M} \mathscr{C} \xrightarrow{F} \mathscr{C}' \Rightarrow M')$,

which is seen to be exactly the data of the lax coslice when M is restricted to functors preserving the exponentiable objects in $\mathscr{L}$. One may describe this subcategory more elegantly by a generalisation of a comma category in which morphisms between objects are taken in another category, though we shall not spell this out here. □

Proposition A.15. *Let $\mathscr{C}$ be a category with finite colimits and let $f : X \to Y$ be a morphism in $\mathscr{C}$. The pushout functor $Y +_X (-) : X/\mathscr{C} \to Y/\mathscr{C}$ is left adjoint to the precomposition functor $(-) \circ f : Y/\mathscr{C} \to X/\mathscr{C}$.*

Proof. Let $a : Y \to A$ and $b : X \to B$ in $\mathscr{C}$. $Y/\mathscr{C}(Y +_X b, a) \cong X/\mathscr{C}(b, a \circ f)$, directly by the universal property of colimits. □

Corollary A.16. *Let $\mathscr{C}$ be a category with binary coproducts and let $X \in \mathscr{C}$. The forgetful functor $U : X/\mathscr{C} \to \mathscr{C}$ has a left adjoint given by $\iota_1 : X \to X + (-)$.* □

Theorem 5.9. *Let $n \in \mathbb{N}_\omega$. The forgetful functor $\lfloor - \rfloor \circ U : L/\mathbf{Law}_{n+1}(S) \to \mathbf{Law}_{n+1}(S) \to \mathbf{Law}_n(S)$ has a left adjoint, sending L' to $L + \lceil L' \rceil$.*

Proof. $\mathbf{Law}_n(S)$ has binary coproducts by Theorem 7.5 and so U has a left adjoint by Corollary A.16. The result then follows from Corollary A.13. □

Theorem 5.10. *Let $F : L \to L'$ be a map of S-sorted n^{th}-order algebraic theories. The functor $L'/\mathbf{Law}_n(S) \to L/\mathbf{Law}_n(S)$ taking a strict model for L' to its precomposition by F has a left adjoint.*

Proof. Direct by Proposition A.15. □

Proposition 5.11. *Let $L : \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. For every model $M : \mathscr{L} \to \mathscr{C}$, there exists an n^{th}-order algebraic theory $L_M : \mathbb{L}_n(S) \to \mathscr{L}_M$, a map $F_M : L \to L_M$ and a fully faithful functor $M' : \mathscr{L}_M \to \mathscr{C}$ such that $M' \circ F_M \cong M$.*

Proof. The identity-on-objects/fully-faithful factorisation system on **Cat** lifts to **CCC**, and so M factorises as $M' \circ F_M$. □

A.5.4 Term algebras.

Lemma 5.14. *For each $n \in \mathbb{N}_\omega$, the semantics functor* $\mathrm{TmAlg} : \mathbf{Law}_{n+1}(S)^{\mathrm{op}} \to \mathbf{CAT}/\mathbf{Law}_n(S)$ *is fully faithful. In particular, it reflects isomorphisms.*

Proof. Lemma 6.4 defines a fully faithful functor $\mathrm{Alg} : \mathbf{RMnd}(p)^{\mathrm{op}} \to \mathbf{CAT}/\mathbf{Law}_n(S)$. As part of the monad correspondence (Theorem 8.2) we also have a fully faithful functor $\mathbf{Law}_{n+1}(S) \to \mathbf{RMnd}(p)$. Since this preserves algebras, the composition

$$\mathbf{Law}_{n+1}(S)^{\mathrm{op}} \to \mathbf{RMnd}(p)^{\mathrm{op}} \xrightarrow{\mathrm{Alg}} \mathbf{CAT}/\mathbf{Law}_n(S)$$

is naturally isomorphic to the semantics functor TmAlg, which is therefore fully faithful. □

Lemma A.17. *If $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ is an $(n+1)^{th}$-order algebraic theory and $A : \mathscr{L} \to \mathbf{Set}$ is cartesian, then $L_A : \mathbb{L}_n(S) \to \mathscr{L}_A$ (defined in Section 5.4) is an n^{th}-order algebraic theory. Moreover, L_A is unique (up to isomorphism in $\mathbf{Law}_n(S)$ such that there exists a natural isomorphism $A \circ L \cong \lceil \mathscr{L}_A \rceil(1, \lceil L_A \rceil(-))$).*

Proof. First note that if $A \circ L \cong \lceil \mathscr{L}_A \rceil(1, \lceil L_A \rceil(-))$, then since $L' \mapsto \lceil \mathscr{L}' \rceil(1, \lceil L \rceil(-))$ is fully faithful (Lemma A.12), L_A is the unique functor with this property. It remains for us to show that L_A is an n^{th}-order algebraic theory, and that this natural isomorphism exists.

For $n = 0$, we trivially have $L_A = s \mapsto A(s) \in \mathbf{Set}^S$. Each first-order arity has the form $\prod_i s_i$, and we have isomorphisms,

$$\begin{aligned}
A(L(\textstyle\prod_i s_i)) &= A(\textstyle\prod_i s_i) && (L \text{ is identity-on-objects})\\
&\cong \textstyle\prod_i A(s_i) && (A \text{ preserves products})\\
&= \textstyle\prod_i L_A(s_i) && (\text{definition of } L_A)\\
&\cong \textstyle\prod_i \lceil \mathscr{L}_A \rceil(1, s_i) && (\text{coreflection})\\
&\cong \lceil \mathscr{L}_A \rceil(1, \textstyle\prod_i s_i) && (\text{products in } \lceil \mathscr{L}_A \rceil)\\
&= \lceil \mathscr{L}_A \rceil(1, \lceil L_A \rceil(\textstyle\prod_i s_i)) && (\lceil L_A \rceil \text{ is identity-on-objects})
\end{aligned}$$

which are natural.

For $n > 0$, the universal properties of products and exponentials imply that $\mathscr{L}_A$ is a category and that L_A is an identity-on-objects functor. $\mathscr{L}_A$ has finite products, since:

$$\begin{aligned}
\mathscr{L}_A(X, \textstyle\prod_i Y_i) &= A((\textstyle\prod_i Y_i)^X) && (\text{definition of } \mathscr{L}_A)\\
&\cong A(\textstyle\prod_i (Y_i{}^X)) && ((-)^X \text{ preserves products})\\
&\cong \textstyle\prod_i A(Y_i{}^X) && (A \text{ preserves products})\\
&\cong \textstyle\prod_i \mathscr{L}_A(X, Y_i) && (\text{definition of } \mathscr{L}_A)
\end{aligned}$$

and this sends the identity on $\prod_i Y_i$ to $(L_A(\pi_i))_i$. Exponentials are trivial, and so L_A is an n^{th}-order algebraic theory.

To construct the required family of isomorphisms we note that every $(n+1)^{\text{th}}$-order arity has the form $\prod_i {s_i}^{X_i}$ for $X_i \in \mathbb{L}_n(S)(S)$, and we have isomorphisms,

$$\begin{aligned}
A(L(\textstyle\prod_i {s_i}^{X_i})) &= A(\textstyle\prod_i {s_i}^{X_i}) && (L \text{ is identity-on-objects}) \\
&\cong \textstyle\prod_i A({s_i}^{X_i}) && (A \text{ preserves products}) \\
&\cong \textstyle\prod_i \mathscr{L}_A(X_i, s_i) && (\text{definition of } L_A) \\
&\cong \textstyle\prod_i \lceil \mathscr{L}_A \rceil (X_i, s_i) && (\text{coreflection}) \\
&\cong \lceil \mathscr{L}_A \rceil (1, \textstyle\prod_i {s_i}^{X_i}) && (\text{products and exponentials in } \lceil \mathscr{L}_A \rceil) \\
&= \lceil \mathscr{L}_A \rceil (1, \lceil L_A \rceil (\textstyle\prod_i {s_i}^{X_i})) && (\lceil L_A \rceil \text{ is identity-on-objects})
\end{aligned}$$

which are natural. □

Below we explicate the relationship between term algebras and cartesian functors. Recall that $\mathbf{Cart}(\mathscr{L}, \mathbf{Set})$ is the category of cartesian functors $\mathscr{L} \to \mathbf{Set}$ and natural transformations between them (Definition A.2).

Lemma A.18. *Suppose that $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ is an $(n+1)^{th}$-order algebraic theory, where $n \in \mathbb{N}_\omega$. Using the natural isomorphisms $A \circ L \cong \lceil \mathscr{L}_A \rceil (1, \lceil L_A \rceil (-))$ from Lemma A.17, the following square forms a 2-pullback in* **CAT**.

$$\begin{tikzcd}
\mathbf{Cart}(\mathscr{L}, \mathbf{Set}) \arrow[r, hook] \arrow[d, "L_{(-)}"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\cong"] & {[\mathscr{L}, \mathbf{Set}]} \arrow[d, "- \circ L"] \\
\mathbf{Law}_n(S) \arrow[r, "L' \mapsto \lceil \mathscr{L}' \rceil (1, \lceil L' \rceil (-))"'] & {[\mathbb{L}_{n+1}(S), \mathbf{Set}]}
\end{tikzcd}$$

Proof. The construction of an n^{th}-order algebraic theory L_A from a cartesian functor A is functorial by sending a natural transformation $\alpha : A \Rightarrow A'$ to $L_\alpha : L_A \to L_{A'}$ in $\mathbf{Law}_n(S)$ given on morphisms by

$$L_A(X, Y) = A(Y^X) \xrightarrow{\alpha_{Y^X}} A'(Y^X) = L'_A(X, Y)$$

This is a morphism in $\mathbf{Law}_n(S)$ by naturality of α.

To verify that we have a 2-pullback, we use the universal property described in Saville [2019, Lemma 7.3.6]. Suppose we have a category $\mathscr{C}$ and functors $F : \mathscr{C} \to \mathbf{Law}_n(S)$ and $G : \mathscr{C} \to [\mathscr{L}, \mathbf{Set}]$, together with natural isomorphisms $G(V) \circ L \cong \mathrm{cod}(\lceil F(V) \rceil)(1, \lceil F(V) \rceil (-))$, natural in $V \in \mathscr{C}$. $G(V)$ is a cartesian functor, since $\mathrm{cod}(\lceil F(V) \rceil)(1, \lceil F(V) \rceil (-))$ is cartesian, and $G(V)$ together with L forms the identity-on-objects/fully-faithful factorisation in **Cart**, so G factors through $\mathbf{Cart}(\mathscr{L}, \mathbf{Set})$. It now remains to show that the compositions of G with the projections are naturally isomorphic to F and G respectively, as in the following diagram, and that this choice is unique.

$$\begin{tikzcd}
\mathscr{C} \arrow[dr, "G"] \arrow[drr, bend left, "G"] \arrow[ddr, bend right, "F"'] & & \\
& \mathbf{Cart}(\mathscr{L}, \mathbf{Set}) \arrow[r, hook] \arrow[d, "L_{(-)}"'] \arrow[dr, phantom, "\cong"] & {[\mathscr{L}, \mathbf{Set}]} \arrow[d, "- \circ L"] \\
& \mathbf{Law}_n(S) \arrow[r, "L' \mapsto \lceil \mathscr{L}' \rceil (1, \lceil L' \rceil (-))"'] & {[\mathbb{L}_{n+1}, \mathbf{Set}]}
\end{tikzcd}$$

Since $L' \mapsto \lceil \mathscr{L}' \rceil (1, \lceil L' \rceil (-))$ is fully faithful (Lemma A.12), to define a natural isomorphism as on the left in the diagram above, it suffices to find natural isomorphisms $\lceil \mathscr{L}_{G(V)} \rceil (1, \lceil L_{G(V)} \rceil (-)) \cong$

$\mathrm{cod}(\lceil F(V) \rceil)(1, \lceil F(V) \rceil(-))$: these are given by

$$\lceil \mathscr{L}_{G(V)} \rceil(1, \lceil L_{G(V)} \rceil(-)) \;\cong\; G(V) \circ L \;\cong\; \mathrm{cod}(\lceil F(V) \rceil)(1, \lceil F(V) \rceil(-))$$

where the first isomorphism follows from Lemma A.17.

For uniqueness, suppose we have a functor $H : \mathscr{C} \to \mathbf{Cart}(\mathscr{L}, \mathbf{Set})$ and natural transformations ζ_1 and ζ_2 as in the following diagram.

$$\begin{tikzcd}
\mathscr{C} \arrow[dr, "H"] \arrow[drr, bend left, "G"] \arrow[ddr, bend right, "F"'] & & \\
& \mathbf{Cart}(\mathscr{L}, \mathbf{Set}) \arrow[u, phantom, "\Uparrow \zeta_1"] \arrow[dl, phantom, "\Leftarrow \zeta_2"] \arrow[r, hook] \arrow[d, "L_{(-)}"] \arrow[dr, phantom, "\cong"] & {[\mathscr{L}, \mathbf{Set}]} \arrow[d, "- \circ L"] \\
& \mathbf{Law}_n(S) \arrow[r, "L' \mapsto \lceil \mathscr{L}' \rceil(1, \lceil L \rceil(-))"'] & {[\mathbb{L}_{n+1}, \mathbf{Set}]}
\end{tikzcd}$$

ζ_1 forms a natural transformation $H \Rightarrow G$, which we use as the unique 2-cell exhibiting G as satisfying the universal property. To show that

$$\begin{tikzcd}
L_{H(V)} \arrow[rr, "L_{\zeta_1}"] \arrow[dr, "\zeta_2"'] & & L_{G(V)} \arrow[dl, "\cong"] \\
& F(V) &
\end{tikzcd}$$

commutes, it suffices for the following to commute, because $L' \mapsto \lceil \mathscr{L}' \rceil(1, \lceil L' \rceil(-))$ is faithful (Lemma A.12).

$$\begin{tikzcd}
\lceil \mathscr{L}_{H(V)} \rceil(1, \lceil L_{H(V)} \rceil(-)) \arrow[r, "L_{\lceil \zeta_1 \rceil}"] \arrow[d, "\lceil \zeta_2 \rceil"'] & \lceil \mathscr{L}_{G(V)} \rceil(1, \lceil L_{G(V)} \rceil(-)) \arrow[d, "\cong"] \\
\lceil F(V) \rceil(1, -) & G(V) \circ L \arrow[l, "\cong"]
\end{tikzcd}$$

This holds because ζ_1 and ζ_2 form a fill-in. Finally, applying the universal property to the fill-in given by G clearly yields a natural isomorphism. □

Proposition 5.15. *Let $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{th}$-order algebraic theory, where $n \in \mathbb{N}_\omega$. Then L-*$\mathbf{TmAlg}$ *is equivalent to* $\mathbf{Cart}(\mathscr{L}, \mathbf{Set})$*, and this equivalence commutes up to natural isomorphism with the forgetful functors into* $\mathbf{Law}_n(S)$.

Proof. The functor $- \circ L$ is a discrete isofibration by Bourke and Garner [2019, Example 3], so the pullback defining L-$\mathbf{TmAlg}$ forms a 2-pullback by Joyal and Street [1993, Corollary 1]. The required equivalence then exists because the cartesian functors also form a 2-pullback (Lemma A.18). □

Proposition 5.16. *Let $L : \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. Up to the equivalence of Proposition 5.15, the hom-functor $\mathscr{L}(1, -) : \mathscr{L} \to$* $\mathbf{Set}$ *is the initial term algebra.*

Proof. Let $T : \mathscr{L} \to \mathbf{Set}$ be a term algebra. $[\mathscr{L}, \mathbf{Set}](\mathscr{L}(1, -), T) \cong T(1) \cong 1$ by the Yoneda lemma and that T is cartesian, so there is a unique morphism from $\mathscr{L}(1, -)$ to T. □

Lemma 5.17. *Let $L : \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. For every term algebra $A : \mathscr{L} \to$* $\mathbf{Set}$*, there exists an n^{th}-order algebraic theory $L_A : \mathbb{L}_n(S) \to \mathscr{L}_A$ and a map $F_A : L \to L_A$ such that $\mathscr{L}_A(1, F_A(-)) \cong A$.*

Proof. We prove this as an intermediate result of Proposition 5.18. □

Proposition 5.18. *L-$\mathbf{TmAlg}$ is a coreflective subcategory of $L/\mathbf{Law}_{n+1}(S)$, for $n \in \mathbb{N}_\omega$.*

Proof. We define a functor $L/\mathbf{Law}_{n+1}(S) \to L$-$\mathbf{TmAlg}$. Given a coslice $(L', F : L \to L')$, the following diagram commutes up to a natural isomorphism given by the counit $\varepsilon_{\mathscr{L}'}$ of the coreflection.

$$\begin{tikzcd}
& \mathbf{Set} & \\
\mathscr{L}' \arrow[ur, "\mathscr{L}'(1,-)"] & & \lceil \lfloor \mathscr{L}' \rfloor \rceil \arrow[ul, "\lceil \lfloor \mathscr{L}' \rfloor \rceil(1,-)"'] \arrow[ll, "\varepsilon_{\mathscr{L}'}"] \\
\mathscr{L} \arrow[u, "F"] & & \\
\mathbb{L}_{n+1}(S) \arrow[u, "L"] \arrow[rr, equal] & & \mathbb{L}_{n+1}(S) \arrow[uull, "L'"] \arrow[uu, "\lceil \lfloor L' \rfloor \rceil"']
\end{tikzcd}$$

Since $- \circ L$ is an discrete isofibration, there therefore exists a term algebra $(\lfloor L' \rfloor, A)$ with $A \cong \mathscr{L}'(1, F(-))$, which we take to be the image of $(L', F : L \to L')$ under the functor $L/\mathbf{Law}_{n+1}(S) \to L$-$\mathbf{TmAlg}$.

In the other direction, we define a functor L-$\mathbf{TmAlg} \to L/\mathbf{Law}_{n+1}(S)$ as follows. Consider a term algebra (L', A). For all $X \in \mathbb{L}_{n+1}(S)$, we have $\lceil \mathscr{L}' \rceil(1, X) = A(X)$ by the pullback condition. Hence, for all $X, Y \in \mathbb{L}_n(S)$, we have $\mathscr{L}'(X, Y) = A(Y^X)$, because $\lceil \mathscr{L}' \rceil(1, -)$ is fully faithful (Lemma A.12) and $\mathscr{L}'$ has strict exponentials. We define an $(n+1)^{\text{th}}$-order algebraic theory $L'' : \mathbb{L}_{n+1}(S) \to \mathscr{L}''$, for which $\mathscr{L}''$ is a full supercategory of $\mathscr{L}$ and such that, for every $f : X \to Y$ for $X \in \mathbb{L}_{n+1}(S)$, $Y \in \mathbb{L}_{n+1}(S)$, where $X \notin \mathbb{L}_n(S)$, we have a unique morphism $\bar{f} \in \mathscr{L}''(X, Y)$ satisfying $\bar{f} \circ x = A(f)(x)$ for every $x \in A(X)$. We also define a map of $(n+1)^{\text{th}}$-order algebraic theories $L \to L''$ sending each $f : X \to Y$ in $\lfloor \mathscr{L} \rfloor$ to $F(\lambda f) \circ \langle\rangle^{-1} : 1 \to F(1) \to F(Y^X) = \mathscr{L}''(X, Y)$, and each $f : X \to Y$ in $\mathscr{L}$ for $X \notin \mathbb{L}_{n+1}(S)$ to $\bar{f}$.

Consider a term algebra (L', A). The image of both L' and A under the composite $\lceil L \rceil$-$\mathbf{TmAlg} \to L/\mathbf{Law}_{n+1}(S) \to \lceil L \rceil$-$\mathbf{TmAlg}$ is naturally isomorphic to the identity, the former because $\lfloor \mathscr{L}'' \rfloor = \mathscr{L}'$ and the latter because maps of theories are identity-on-objects, so $\mathscr{L}''(1, F(X)) = \mathscr{L}''(1, X) = A(X)$, with the image of morphisms $f : X \to Y$ coinciding by construction.

Functoriality is straightforward. Hence the functor $\lceil L \rceil$-$\mathbf{TmAlg} \to L/\mathbf{Law}_{n+1}(S)$ is a section up to isomorphism, and so $\lceil L \rceil$-$\mathbf{TmAlg}$ is a coreflective subcategory of $L/\mathbf{Law}_{n+1}(S)$. □

Proposition 5.20. *$L/\mathbf{Law}_n(S)$ is a coreflective subcategory of $\lceil L \rceil$-$\mathbf{TmAlg}$, for $n \in \mathbb{N}_\omega$.*

Proof. The proof is similar to that of Proposition 5.18, though somewhat simpler.

We define a functor $L/\mathbf{Law}_n(S) \to \lceil L \rceil$-$\mathbf{TmAlg}$ mapping a coslice $(L', F : L \to L')$ to the pair $(L', \lceil \mathscr{L}' \rceil(1, \lceil F \rceil(-)))$, which is straightforwardly a term algebra for $\lceil L \rceil$.

In the other direction, we define a functor $\lceil L \rceil$-$\mathbf{TmAlg} \to L/\mathbf{Law}_n(S)$ as follows. Consider a term algebra (L', A). For all $X \in \mathbb{L}_{n+1}(S)$, we have $\lceil \mathscr{L}' \rceil(1, X) = A(X)$ by the pullback condition. Hence, for all $X, Y \in \mathbb{L}_n(S)$, we have $\mathscr{L}'(X, Y) = A(Y^X)$, because $\lceil \mathscr{L}' \rceil(1, -)$ is fully faithful (Lemma A.12) and $\mathscr{L}'$ has strict exponentials. We define a map of n^{th}-order algebraic theories $L \to L'$ sending each $f : X \to Y$ in $\mathscr{L}$ to $A(\lambda f) \circ \langle\rangle^{-1} : 1 \to A(1) \to A(Y^X) = \mathscr{L}'(X, Y)$.

Consider a coslice $F : L \to L'$. The image under the composite $L/\mathbf{Law}_n(S) \to \lceil L \rceil$-$\mathbf{TmAlg} \to L/\mathbf{Law}_n(S)$ on L' is the identity. Consider a morphism $f : X \to Y$ in $\mathscr{L}$. It is mapped under the image on the map F to

$$\begin{aligned}
\lceil\mathscr{L}'\rceil(1, \lceil F\rceil(\lambda f)) &= \lceil\mathscr{L}'\rceil(1, \lambda(\lceil F\rceil(f))) && \text{maps strictly preserve exponentials} \\
&\cong \lceil\mathscr{L}'\rceil(1, \lambda(F(f))) && \text{counit of coreflection is invertible} \\
&\cong F(f) && \text{uncurrying}
\end{aligned}$$

Functoriality is straightforward. Hence the functor $L/\mathbf{Law}_n(S) \to \lceil L\rceil\text{-}\mathbf{TmAlg}$ is a section up to isomorphism, and so $L/\mathbf{Law}_n(S)$ is a coreflective subcategory of $\lceil L\rceil\text{-}\mathbf{TmAlg}$. □

Corollary A.19. *For every $L : \mathbb{L}_\omega(S) \to \mathscr{L}$, there is an equivalence $L/\mathbf{Law}_\omega(S) \simeq L\text{-}\mathbf{TmAlg}$.*

Proof. It suffices to show that the counit of the coreflective adjunction of Proposition 5.18, in the case $n = \omega$, is an isomorphism. Let $L' : \mathbb{L}_\omega(S) \to \mathscr{L}'$ with $F : L \to L'$ be a coslice under L. The term algebra $A_L : \mathscr{L} \to \mathbf{Set}$ is given by $\mathscr{L}'(1, F(-))$. For all $X \in \mathbb{L}_\omega(S)$, we have $\mathscr{L}_{A_L}(1, X) \overset{\text{def}}{=} A_{L'}(X) \overset{\text{def}}{=} \mathscr{L}'(1, F(X)) = \mathscr{L}'(1, X)$ as F is identity-on-objects. As every object in $\mathbb{L}_\omega(S)$ is exponentiable, it follows that, for all $X, Y \in \mathbb{L}_\omega(S)$, we have $\mathscr{L}_{A_L}(X, Y) \cong \mathscr{L}_{A_L}(1, Y^X) = \mathscr{L}(1, Y^X) \cong \mathscr{L}(X, Y)$. Thus, as maps of ω-order algebraic theories are identity-on-objects, $\mathscr{L}_{A_L} \cong L$, from which the equivalence follows. □

A.5.5 Theories from arities.

Lemma A.20. *Suppose that $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ is an $(n+1)^{th}$-order algebraic theory, where $n \in \mathbb{N}_\omega$. The simple slices of L form a functor $L/\!\!/- : \mathscr{L}^{\mathrm{op}} \to \mathbf{Law}_{n+1}(S)$.*

Proof. That $\mathscr{L}/\!\!/X$ is a category follows from its characterisation as a Kleisli category; functoriality of $L/\!\!/X : \mathbb{L}_{n+1}(S) \to \mathscr{L}/\!\!/X$ is clear. The projection and evaluation morphisms in $\mathbb{L}_{n+1}(S)$ under $L/\!\!/X$

$$(L/\!\!/X)(\pi_j) \in (\mathscr{L}/\!\!/X)(\textstyle\prod_i Y_i, Y_j)$$
$$(L/\!\!/X)(\mathrm{ev}_{Y',Y}) \in (\mathscr{L}/\!\!/X)(Y^{Y'} \times Y', Y)$$

satisfy the universal properties of projections and evaluations in $\mathscr{L}/\!\!/X$. Hence $L/\!\!/X$ is an $(n+1)^{\mathrm{th}}$-order algebraic theory.

For each $f : X' \to X$ in $\mathscr{L}$, the identity-on-objects functor $(L/\!\!/f) : (L/\!\!/X) \to (L/\!\!/X')$ is defined on morphisms by

$$(\mathscr{L}/\!\!/X)(Y, Y') \;=\; \mathscr{L}(X \times Y, Y') \xrightarrow{-\circ(f\times Y)} \mathscr{L}(X' \times Y, Y') \;=\; (\mathscr{L}/\!\!/X')(Y, Y')$$

For each $g : Y \to Y'$ in $\mathbb{L}_{n+1}(S)$, we have $(L/\!\!/f)((L/\!\!/X')(g)) = (L/\!\!/X)(g)$, so $L/\!\!/f$ is a map of algebraic theories. Finally, it is clear that $L/\!\!/-$ preserves identities and composition, so is a functor $\mathscr{L}^{\mathrm{op}} \to \mathbf{Law}_n(S)$. □

To exhibit the universal property of p, we first show that the simple slices $L/\!\!/-$ satisfy a similar universal property. For each $X \in \mathbb{L}_{n+1}(S)$, we write $X^* : L \to L/\!\!/X$ for the canonical morphism of $(n+1)^{\mathrm{th}}$-order algebraic theories sending $f : Y \to Y'$ to $f \circ \pi_Y : X \times Y \to Y'$.

Lemma A.21. *Suppose that $X \in \mathbb{L}_{n+1}(S)$, where $n \in \mathbb{N}_\omega$, and that $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ is an $(n+1)^{th}$-order algebraic theory. For each n^{th}-order algebraic theory $L' : \mathbb{L}_n(S) \to \mathscr{L}'$, morphism $F : L \to L'$ in $\mathbf{Law}_n(S)$, and $x \in \mathscr{L}'(1, X)$, there is a unique morphism $\widehat{(F, x)} : L/\!\!/X \to L'$ in $\mathbf{Law}_n(S)$ such that $\widehat{(F, x)} \circ X^* = F$ and $\widehat{(F, x)}(\pi_X) = x \in \mathscr{L}'(1, X)$.*

Proof. We first show uniqueness. If $\widehat{(F,x)}(\pi_X) = x$ then

$$\widehat{(F,x)}(\langle \pi_X, \pi_Y \rangle) = \langle \widehat{(F,x)}(\pi_X), \widehat{(F,x)}(\pi_Y) \rangle = \langle x, \mathrm{id}_Y \rangle \ \in \ \mathscr{L}'(Y, X \times Y)$$

where the first equality uses that $\widehat{(F,x)}$ preserves products (tupling in $\mathscr{L} /\!\!/ X$ is the same as tupling in $\mathscr{L}$), and the second equality uses that $\widehat{(F,x)}$ preserves identities (the identity on Y in $\mathscr{L} /\!\!/ X$ is π_Y). Now, if $\widehat{(F,x)} \circ X^* = F$, then, for all $f \in \mathscr{L}(X \times Y, Y')$,

$$\begin{aligned}
\widehat{(F,x)}(f) &= \widehat{(F,x)}(X^*(f) \circ \langle \pi_X, \langle \pi_X, \pi_Y \rangle \rangle) && \text{(definition of } X^*\text{, products in } \mathscr{L}) \\
&= \widehat{(F,x)}(X^*(f)) \circ \widehat{(F,x)}\, \langle \pi_X, \pi_Y \rangle && (\widehat{(F,x)} \text{ preserves composition}) \\
&= \widehat{(F,x)}(X^*(f)) \circ \langle x, \mathrm{id}_Y \rangle && \text{(above)} \\
&= F(f) \circ \langle x, \mathrm{id}_Y \rangle && (\widehat{(F,x)} \circ X^* = F)
\end{aligned}$$

which implies uniqueness.

It remains to show existence. Define the identity-on-objects functor $\widehat{(F,x)}$ by $\widehat{(F,x)}(f) \stackrel{\text{def}}{=} F(f) \circ \langle x, \mathrm{id}_Y \rangle$. We have $\widehat{(F,x)}\pi_X = x$ and $\widehat{(F,x)} \circ X^* = F$ because F preserves products. The latter implies

$$\widehat{(F,x)} \circ (L /\!\!/ X) = \widehat{(F,x)} \circ X^* \circ L = F \circ L = L'$$

and this implies preservation of identities. Preservation of composition holds because

$$\begin{aligned}
\widehat{(F,x)}(g \circ \langle \pi_X, f \rangle) &= F(g \circ \langle \pi_X, f \rangle) \circ \langle x, \mathrm{id}_Y \rangle && \text{(definition of } \widehat{(F,x)}) \\
&= Fg \circ \langle \pi_X, Ff \rangle \circ \langle x, \mathrm{id}_Y \rangle && (F \text{ preserves composition and products}) \\
&= Fg \circ \langle x, \mathrm{id}_{Y'} \rangle \circ Ff \circ \langle x, \mathrm{id}_Y \rangle && \text{(products in } \mathscr{L}') \\
&= \widehat{(F,x)}g \circ \widehat{(F,x)}f && \text{(definition of } \widehat{(F,x)})
\end{aligned}$$

So $\widehat{(F,x)}$ is a map of algebraic theories. □

We specialise this universal property to $L = \mathrm{Id}$.

Corollary A.22. *Suppose that $X \in \mathbb{L}_{n+1}(S)$, where $n \in \mathbb{N}_\omega$. For each $(n+1)^{th}$-order algebraic theory $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ and $x \in \mathscr{L}(1, X)$, there is a unique morphism $\hat{x} : \mathrm{Id} /\!\!/ X \to L$ in $\mathbf{Law}_{n+1}(S)$ such that $\hat{x}(\pi_X) = x$. In particular, we have bijections*

$$\mathscr{L}(1, X) \ \cong \ \mathbf{Law}_{n+1}(S)(\mathrm{Id} /\!\!/ X, L)$$

natural in X and L.

Proof. Since $\mathrm{Id} \in \mathbf{Law}_{n+1}(S)$ is initial, it follows from Lemma A.21 that we have $\hat{x} = \widehat{(L,x)}$. Naturality is simple to check. □

Lemma A.23. *The component of the counit $\lceil \lfloor \mathrm{Id} /\!\!/ X \rfloor \rceil \to \mathrm{Id} /\!\!/ X$ of the adjunction $\lceil - \rceil \dashv \lfloor - \rfloor$ on $\mathrm{Id} /\!\!/ X$ is an isomorphism for each $n \in \mathbb{N}_\omega$ and $X \in \mathbb{L}_{n+1}(S)$.*

Proof. We first consider $n > 0$. Morphisms $Y \to Z$, for $Y, Z \in \mathbb{L}_n(S)$, in $\lceil \lfloor \mathrm{Id} /\!\!/ X \rfloor \rceil$ are given by morphisms $Y \to Z$ in $\mathrm{Id} /\!\!/ X$, so we shall only have to consider the case $Y, Z \notin \mathbb{L}_n(S)$. In fact, it suffices to show that projections of X from Y can be recovered in $\lceil \lfloor \mathrm{Id} /\!\!/ X \rfloor \rceil$, as per the definition of $\mathbb{L}_{n+1}(S)$ in Definition 4.2; these are the only morphisms in $\mathrm{Id} /\!\!/ X$ that are not automatically preserved by forgetting the $(n+1)^{\text{th}}$-order structure. The following diagram commutes in $\mathrm{Id}/\!\!/X$.

$$\begin{tikzcd} Y \arrow[rr, "\pi_X"] \arrow[dr, "!"'] & & X \\ & 1 \arrow[ur, "\pi_X"'] & \end{tikzcd}$$

When $X \in \mathbb{L}_n(S)$, we can therefore recover every projection of X from Y by a corresponding global element $1 \to X$ given by π_X in $\mathrm{Id}/\!\!/X$. However, if $X \notin \mathbb{L}_n(S)$, these global elements do not exist in $\lfloor \mathrm{Id}/\!\!/X \rfloor$. Let $X = {X_2}^{X_1}$ for some $X_1, X_2 \in \mathbb{L}_n(S)$. A projection of X from Y in $\mathrm{Id}/\!\!/X$ is given by the evaluation $Y \times X \times X_1 = Y \times X_2^{X_1} \times X_1 \to X_2$ in $\mathbb{L}_{n+1}(S)$: this is a morphism $Y \times X_1 \to X_2$ in $\lfloor \mathrm{Id}/\!\!/X \rfloor$. Therefore, after applying $\lceil - \rceil$, the global element $1 \to X$ is recovered, and so every projection of X in $\mathrm{Id}/\!\!/X$ exists in $\lceil \lfloor \mathrm{Id}/\!\!/X \rfloor \rceil$. Hence $\lceil \lfloor \mathrm{Id}/\!\!/X \rfloor \rceil \cong \mathrm{Id}/\!\!/X$, which is trivially given by the counit.

Finally, we consider $n = 0$. We only have to consider the morphisms $Y_1 \times \cdots \times Y_k \to X$, as the others are clearly preserved by the counit. By the universal property of cartesian products, it suffices to consider $X = A$ for some $A \in S$. However, as $A \in \mathbb{L}_0(S)$, such a morphism is equivalently given by a corresponding global element $1 \to X$ in $\mathrm{Id}/\!\!/X$, which is preserved by $\lfloor - \rfloor$. Hence $\lceil \lfloor \mathrm{Id}/\!\!/X \rfloor \rceil \cong \mathrm{Id}/\!\!/X$, which again is trivially given by the counit. □

Lemma 5.22. *Let $n \in \mathbb{N}_\omega$ and $X \in \mathbb{L}_{n+1}(S)$. For each n^{th}-order algebraic theory $L : \mathbb{L}_n(S) \to \mathscr{L}$ and $x \in \lceil \mathscr{L} \rceil(1, X)$, there is a unique morphism $F : p(X) \to L$ in $\mathbf{Law}_n(S)$ such that $\lceil F \rceil(\rho_X) = x$. In particular, we have the following bijection, natural in X and L.*

$$\lceil \mathscr{L} \rceil(1, \lceil L \rceil(X)) \;\cong\; \mathbf{Law}_n(S)(p(X), L)$$

Proof. By Corollary A.22, there is a unique morphism $\hat{x} : \mathrm{Id}/\!\!/X \to \lceil L \rceil$ in $\mathbf{Law}_{n+1}(S)$ such that $\hat{x}(\pi_X) = x$. We write $\varepsilon : \lceil \lfloor - \rfloor \rceil \Rightarrow \mathrm{Id}$ for the counit of the coreflection. Then $\varepsilon_{\mathrm{Id}/\!\!/X}$ has an inverse $\varepsilon^{-1}_{\mathrm{Id}/\!\!/X}$ by Lemma A.23, and $\rho_X = \varepsilon^{-1}_{\mathrm{Id}/\!\!/X}(\pi_X)$. We have

$$\begin{aligned}
\lceil F \rceil(\rho_X) = x &\iff \lceil F \rceil \circ \varepsilon^{-1}_{\mathrm{Id}/\!\!/X} = \hat{x} && \text{(uniqueness of } \hat{x}\text{)} \\
&\iff \lceil F \rceil = \hat{x} \circ \varepsilon_{\mathrm{Id}/\!\!/X} && (\varepsilon_{\mathrm{Id}/\!\!/X} \text{ is invertible}) \\
&\iff \lceil F \rceil = \varepsilon_{\lceil L \rceil} \circ \lceil \lfloor \hat{x} \rfloor \rceil && (\varepsilon \text{ is natural}) \\
&\iff \lceil F \rceil = \lceil \eta_L^{-1} \rceil \circ \lceil \lfloor \hat{x} \rfloor \rceil && \text{(triangle law, } \eta_L \text{ is invertible)} \\
&\iff F = \eta_L^{-1} \circ \lfloor \hat{x} \rfloor && (\lceil - \rceil \text{ is fully faithful})
\end{aligned}$$

so $F \stackrel{\text{def}}{=} \eta_L^{-1} \circ \lfloor \hat{x} \rfloor$ is the required unique morphism. Naturality follows from naturality of $\widehat{(-)}$ and of η^{-1}. □

The following property of p is needed later.

Lemma A.24. *For each $n \in \mathbb{N}_\omega$, the functor $p^{\mathrm{op}} : \mathbb{L}_{n+1}(S) \to \mathbf{Law}_n(S)^{\mathrm{op}}$ preserves products.*

Proof. We have

$$\begin{aligned}
\mathbf{Law}_n(S)(p(\textstyle\prod_i X_i), L) &\cong \lceil \mathscr{L} \rceil(1, \textstyle\prod_i X_i) && \text{(Lemma 5.22, } \lceil L \rceil \text{ is identity-on-objects)} \\
&\cong \textstyle\prod_i \lceil \mathscr{L} \rceil(1, X_i) && \text{(products in } \lceil \mathscr{L} \rceil\text{)} \\
&\cong \textstyle\prod_i \mathbf{Law}_n(S)(p(X_i), L) && (\lceil L \rceil \text{ is identity-on-objects, Lemma 5.22)}
\end{aligned}$$

and the composition sends F to $(F \circ p(\pi_i))_i$, using the universal property of p. □

A.6 Relative monads and theories

A.6.1 Preliminaries.

Lemma 6.4. *There is a functor* $\mathrm{Alg} : \mathbf{RMnd}(p)^{\mathrm{op}} \to \mathbf{CAT}/\mathscr{C}$ *that assigns to each relative monad* T *on* $p : \mathscr{C}' \to \mathscr{C}$ *the forgetful functor* $T\text{-}\mathbf{Alg} \to \mathscr{C}$ *from its category of algebras. Moreover,* Alg *is fully faithful, and in particular reflects isomorphisms.*

Proof. Suppose that $m : T \Rightarrow T'$ is a relative monad morphism and $(A, (-)^{\ddagger})$ is a T'-algebra. Then $(A, (-)^{\ddagger} \circ m)$ is an T-algebra because $f^{\ddagger} \circ m \circ \eta = f^{\ddagger} \circ \eta' = f$ and $f^{\ddagger} \circ m \circ g^{\dagger} = f^{\ddagger} \circ (m \circ g)^{\dagger} \circ m = (f^{\ddagger} \circ m \circ g)^{\ddagger} \circ m$. Any T'-algebra homomorphism becomes a T-algebra homomorphism trivially, hence we have a functor $\mathbf{RMnd}(p)^{\mathrm{op}} \to \mathbf{CAT}/\mathscr{C}$.

To show it is fully faithful, let $F : T'\text{-}\mathbf{Alg} \to T\text{-}\mathbf{Alg}$ be a functor over $\mathscr{C}$. For each $X \in \mathscr{C}'$ we have a free T'-algebra $(T'(X), (-)^{\dagger})$: this is sent by F to a T-algebra of the form $(T'(X), (-)^{\ddagger})$, hence we have $m_X \stackrel{\mathrm{def}}{=} {\eta'_X}^{\ddagger} : T(X) \to T'(X)$. This is a relative monad morphism because the Kleisli extension of any morphism $p(X) \to T'(Y)$ is a homomorphism between free T'-algebras, hence a homomorphism between the induced T-algebras. So we have

$$
\begin{aligned}
(m_Y \circ f)^{\dagger} \circ m_X &= ((m_Y \circ f)^{\dagger} \circ \eta'_X)^{\ddagger} && ((m_Y \circ f)^{\dagger} \text{ is an algebra homomorphism}) \\
&= (m_Y \circ f)^{\ddagger} && (\text{left unit law}) \\
&= m_Y \circ f^{\dagger} && (\text{algebra law})
\end{aligned}
$$

and $m_X \circ \eta_X = \eta'_X$ because of the left unit law.

The functor $T'\text{-}\mathbf{Alg} \to T\text{-}\mathbf{Alg}$ induced by this m sends $(A, (-)^{\ddagger})$ to $(A, (-)^{\ddagger} \circ m)$. Given any $f : p(X) \to A$, the extension $f^{\ddagger} : T'(X) \to A$ is a homomorphism from the free T'-algebra, hence a homomorphism from the induced T-algebra. So

$$
\begin{aligned}
f^{\ddagger} \circ m_X &= (f^{\ddagger} \circ \eta'_X)^{\ddagger} && (f^{\ddagger} \text{ is an algebra homomorphism}) \\
&= f^{\ddagger} && (\text{algebra law})
\end{aligned}
$$

and hence the induced functor is the same as F.

Given a relative monad morphism $m : T \Rightarrow T'$, the induced functor between categories of algebras is given by $(A, (-)^{\ddagger}) \mapsto (A, (-)^{\ddagger} \circ m)$, and the relative monad morphism induced by this induced functor is ${\eta'_X}^{\dagger} \circ m_X$, which is m_X by the right unit law. Hence the constructions of morphisms are inverses, and Alg is fully faithful. □

A.6.2 Relative monads from theories.

The bijections (12) in Section 6.2 are given by

$$
\begin{aligned}
\mathbf{Law}_n(S)(p(Y), T_L(X)) &\cong \mathbf{Law}_{n+1}(S)(\lceil p(Y) \rceil, L /\!\!/ X) && (\text{transposing}) \\
&\cong \mathbf{Law}_{n+1}(S)(\mathrm{Id} /\!\!/ Y, L /\!\!/ X) && (\text{Lemma A.23}) \\
&\cong (\mathscr{L} /\!\!/ X)(1, Y) && (\text{Corollary A.22}) \\
&= \mathscr{L}(X, Y) && (\text{simple slice})
\end{aligned}
$$

These are natural in $X \in \mathscr{L}$ and $Y \in \mathbb{L}_{n+1}(S)$ because each isomorphism is. They are also natural in L, i.e. for each $F : L \to L'$ and $G \in \mathbf{Law}_n(S)(p(Y), T_L(X))$, if the bijection sends G to g, then it sends $\lfloor F /\!\!/ X \rfloor \circ G$ to $F(g)$, where $\lfloor F /\!\!/ X \rfloor$ is the restriction of F to $T_L \to T_{L'}$. Since the universal property of p is natural in the algebraic theory, the bijections are equivalently given on $F : p(Y) \to T_L(X)$ by sending $\lceil F \rceil(\rho_Y) \in \lceil T_L(X) \rceil(1, Y)$ along the counit of the coreflection.

Lemma 6.5. *The above defines a functor* $\mathbf{Law}_{n+1}(S) \to \mathbf{RMnd}(p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S))$ *for each* $n \in \mathbb{N}_\omega$. *For each* $(n+1)^{th}$*-order algebraic theory* $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$, *there is an isomorphism of categories* T_L-$\mathbf{Alg} \cong L$-$\mathbf{TmAlg}$ *commuting with the forgetful functors into* $\mathbf{Law}_n(S)$.

Proof. We show that T_L satisfies the three monad laws. As expected, these use unitality and associativity of composition in $\mathscr{L}$. The left unit law $F^\dagger \circ \eta = F$ holds because the bijections (12) are natural in Y. The right unit law ${\eta_X}^\dagger = \mathrm{id}_{T_L(X)}$ is immediate from the definitions. For associativity, consider the following diagram, for $X, X', Y \in \mathbb{L}_{n+1}(S)$.

$$\begin{tikzcd}
\mathbf{Law}_n(S)(p(Y),T_L(X)) \times \mathscr{L}(X',X) \arrow[r, "\mathbf{Law}_n(S)(p(Y),T_L(X))\times\lfloor L/\!\!/-\rfloor"] \arrow[dd, "\cong\times\mathscr{L}(X',X)"'] & \mathbf{Law}_n(S)(p(Y),T_L(X)) \times \mathbf{Law}_n(S)(T_L(X),T_L(X')) \arrow[d, "\circ"] \\
 & \mathbf{Law}_n(S)(p(Y),T_L(X')) \arrow[d, "\cong"] \\
\mathscr{L}(X,Y) \times \mathscr{L}(X',X) \arrow[r, "\circ"] \arrow[d, "\lfloor L/\!\!/-\rfloor\times\lfloor L/\!\!/-\rfloor"'] & \mathscr{L}(X',Y) \arrow[d, "\lfloor L/\!\!/-\rfloor"] \\
\mathbf{Law}_n(S)(T_L(Y),T_L(X)) \times \mathbf{Law}_n(S)(T_L(X),T_L(X')) \arrow[r, "\circ"'] & \mathbf{Law}_n(S)(T_L(Y),T_L(X))
\end{tikzcd}$$

The top commutes by naturality of the bijections (12) in X; the bottom by associativity of composition. After composing with $\mathbf{Law}_n(S)(p(X),T_L(X')) \xrightarrow{\cong} \mathscr{L}(X',X)$ and applying to (F,G), it follows that $G^\dagger \circ F^\dagger = (G^\dagger \circ F)^\dagger$.

Given $F : L \to L'$, let $\lfloor F/\!\!/-\rfloor$ be the restriction of F to $T_L \to T_{L'}$. Then $\lfloor F/\!\!/-\rfloor$ commutes with η by naturality of the bijections (12) in L. Commutativity with $(-)^\dagger$ is expressed by the following diagram.

$$\begin{tikzcd}
\mathbf{Law}_n(S)(p(Y),T_L(X)) \arrow[rr, "\lfloor F/\!\!/X\rfloor\circ-"] \arrow[d, "\cong"'] & & \mathbf{Law}_n(S)(p(Y),T_{L'}(X)) \arrow[d, "\cong"] \\
\mathscr{L}(X,Y) \arrow[rr, "F"'] \arrow[d, "\lfloor L/\!\!/-\rfloor"'] & & \mathscr{L}'(X,Y) \arrow[d, "\lfloor L'/\!\!/-\rfloor"] \\
\mathbf{Law}_n(S)(T_L(Y),T_L(X)) \arrow[dr, "\lfloor F/\!\!/X\rfloor\circ-"'] & & \mathbf{Law}_n(S)(T_{L'}(Y),T_{L'}(X)) \arrow[dl, "-\circ\lfloor F/\!\!/Y\rfloor"] \\
 & \mathbf{Law}_n(S)(T_L(Y),T_{L'}(X)) &
\end{tikzcd}$$

The square commutes by naturality; the pentagon because F is strictly cartesian.

To get the required isomorphism between the categories of algebras, we show that T_L-$\mathbf{Alg}$ is the vertex of the following pullback.

$$\begin{tikzcd}
T_L\text{-}\mathbf{Alg} \arrow[r, "P_L"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\mathscr{L},\mathbf{Set}]} \arrow[d, "-\circ L"] \\
\mathbf{Law}_n(S) \arrow[r, "L'\mapsto\lceil\mathscr{L}'\rceil(1,\lceil L'\rceil(-))"'] & {[\mathbb{L}_{n+1}(S),\mathbf{Set}]}
\end{tikzcd}$$

The unlabelled arrow is the forgetful functor. To define P_L, we note that, for any algebra $(A : \mathbb{L}_n(S) \to \mathscr{A}, (-)^\ddagger)$, the map $X \mapsto \lceil\mathscr{A}\rceil(1, \lceil A\rceil(-))$ extends to a functor $P_L(A(-)^\ddagger) : \mathscr{L} \to \mathbf{Set}$ by sending $e \in \mathscr{L}(X, Y)$ to

$$\begin{aligned}\lceil \mathscr{A} \rceil(1, X) \xrightarrow{\cong} \mathbf{Law}_n(S)(p(X), A) \xrightarrow{(-)^{\ddagger}} \mathbf{Law}_n(S)(T_L(X), A) \\ \xrightarrow{- \circ \lfloor L /\!\!/ e \rfloor \circ \eta_Y} \mathbf{Law}_n(S)(p(Y), A) \xrightarrow{\cong} \lceil \mathscr{A} \rceil(1, Y)\end{aligned}$$

Preservation of identities is immediate from $F^{\ddagger} \circ \eta = F$. For preservation of composition, naturality implies that (12) sends $\lfloor L /\!\!/ e \rfloor \circ \eta$ to e, so $(\lfloor L /\!\!/ e \rfloor \circ \eta)^{\dagger} = \lfloor L /\!\!/ e \rfloor$. Hence

$$\begin{aligned}
(F^{\ddagger} \circ \lfloor L /\!\!/ e \rfloor \circ \eta)^{\ddagger} \circ \lfloor L /\!\!/ e' \rfloor \circ \eta &= F^{\ddagger} \circ (\lfloor L /\!\!/ e \rfloor \circ \eta)^{\dagger} \circ \lfloor L /\!\!/ e' \rfloor \circ \eta && \text{(algebra)} \\
&= F^{\ddagger} \circ \lfloor L /\!\!/ e \rfloor \circ \lfloor L /\!\!/ e' \rfloor \circ \eta && \text{(above)} \\
&= F^{\ddagger} \circ \lfloor L /\!\!/ (e \circ e') \rfloor \circ \eta && \text{(definition of } \lfloor L /\!\!/ - \rfloor)
\end{aligned}$$

On algebra homomorphisms $h : (A, (-)^{\ddagger}) \to (A', (-)^{\ddagger'})$, the natural transformation $P_L(h)$ sends $x \in \lceil \mathscr{A} \rceil(1, X)$ to $\lceil h \rceil(x) \in \lceil \mathscr{A}' \rceil(1, X)$. This completes the definition of the functor P_L.

For commutativity of the pullback square, the only interesting case is for morphisms $e \in \mathbb{L}_{n+1}(S)(X, Y)$. We have

$$\begin{aligned}
(-)^{\ddagger} \circ \lfloor L /\!\!/ L(e) \rfloor \circ \eta &= (-)^{\ddagger} \circ \eta \circ p(e) && \text{(naturality of (12) on both sides)} \\
&= - \circ p(e) && \text{(algebra law)}
\end{aligned}$$

which implies $P_L(A, (-)^{\ddagger})(L(e)) = - \circ \lceil A \rceil(e)$, by naturality of (12).

We now show the square is a pullback. Let $F : \mathscr{C} \to [\mathscr{L}, \mathbf{Set}]$ and $G : \mathscr{C} \to \mathbf{Law}_n(S)$ be functors making the square commute. We show that there is a unique functor $\mathscr{C} \to T_L\text{-}\mathbf{Alg}$ such that the required diagram commutes. Given $V \in \mathscr{C}$, the underlying n^{th}-order algebraic theory of the algebra is necessarily $F(V)$. To define $(-)^{\ddagger}$, note that, for $X, Y \in \mathbb{L}_{n+1}(S)$, we have $G(V)(X) = \lceil F(V) \rceil(1, X)$, and hence $G(V)$ sends each element of $\mathscr{L}(X, Y)$ to a function $\lceil F(V) \rceil(1, X) \to \lceil F(V) \rceil(1, Y)$; this is natural in Y because $G(V)(L(e)) = \lceil F(V) \rceil$. Since $G(V)(e)$ must be equal to

$$\begin{aligned}\lceil F(V) \rceil(1, X) \xrightarrow{\cong} \mathbf{Law}_n(S)(p(X), F(V)) \xrightarrow{(-)^{\ddagger}} \mathbf{Law}_n(S)(T_L(X), F(V)) \\ \xrightarrow{- \circ \lfloor L /\!\!/ e \rfloor \circ \eta} \mathbf{Law}_n(S)(p(Y), F(V)) \xrightarrow{\cong} \lceil F(V) \rceil(1, Y)\end{aligned}$$

and $\lfloor L /\!\!/ e \rfloor \circ \eta$ is the result of sending e along (12), $(-)^{\ddagger}$ is necessarily given by

$$\begin{aligned}
& \mathbf{Law}_n(S)(p(X), F(V)) \\
\cong\ & \lceil F(V) \rceil(1, X) && \text{(Lemma 5.22)} \\
\to\ & [\mathbb{L}_{n+1}(S), \mathbf{Set}](\mathscr{L}(X, L(-)), \lceil F(V) \rceil(1, -)) && (G(V) \text{ on morphisms}) \\
\cong\ & [\mathbb{L}_{n+1}(S), \mathbf{Set}](\mathbf{Law}_n(S)(p(-), \lfloor L /\!\!/ X \rfloor), \mathbf{Law}_n(S)(p(-), F(V))) && \text{((12), Lemma 5.22)} \\
\cong\ & \mathbf{Law}_n(S)(\lfloor L /\!\!/ X \rfloor, F(V)) && \text{(Lemma A.34)}
\end{aligned}$$

We have $f^\ddagger \circ \eta = f$ because η is given by applying (12) to the identity. For the other algebra law, the function

$$\begin{aligned}
&\mathbf{Law}_n(S)(p(Y), T_L(X)) \times \mathbf{Law}_n(S)(p(X), F(V)) \\
&\cong \mathscr{L}(X, Y) \times \lceil F(V) \rceil (1, X) && ((12), \text{Lemma 5.22}) \\
&\to \mathscr{L}(X, Y) \times [\mathbb{L}_{n+1}(S), \mathbf{Set}](\mathscr{L}(X, L(-)), \lceil F(V) \rceil (1, -)) && (G(V) \text{ on morphisms}) \\
&\to [\mathbb{L}_{n+1}(S), \mathbf{Set}](\mathscr{L}(Y, L(-)), \lceil F(V) \rceil (1, -)) && (\text{composition}) \\
&\cong [\mathbb{L}_{n+1}(S), \mathbf{Set}](\mathbf{Law}_n(S)(p(-), T_L(Y)), \mathbf{Law}_n(S)(p(-), F(V))) && ((12), \text{Lemma 5.22}) \\
&\cong \mathbf{Law}_n(S)(T_L(Y), F(V)) && (\text{Lemma A.34})
\end{aligned}$$

is equal to $(f, g) \mapsto g^\ddagger \circ f^\dagger$ and to $(f, g) \mapsto (g^\ddagger \circ f)^\ddagger$ by naturality of (12) in X.

This defines and establishes uniqueness of $\mathscr{C} \to T_L\text{-}\mathbf{Alg}$ on objects. Each morphism $v : V \to V'$ in $\mathscr{C}$ is necessarily sent to $F(v)$ in order to commute with the forgetful functor into $\mathbf{Law}_n(S)$. Since $G(v)_L = \mathrm{cod}(\lceil F(V) \rceil)(1, \lceil F(V) \rceil (-))$, this also commutes with P_L. It is easy to see that this is an algebra homomorphism, using naturality of the universal property of p. □

Lemma A.25. *Suppose that $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ is an $(n+1)^{th}$-order algebraic theory for $n \in \mathbb{N}_\omega$. The bijections (12) form an isomorphism of categories $\mathbf{Kl}(T_L)^{\mathrm{op}} \cong \mathscr{L}$.*

Proof. Immediate from the definitions of the unit and Kleisli extension. □

A.6.3 Theories from relative monads.

Lemma A.26. *If $(T, \eta, (-)^\dagger)$ is a relative monad on $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$, where $n \in \mathbb{N}_\omega$, then $L_T : \mathbb{L}_{n+1}(S) \to \mathbf{Kl}(T)^{\mathrm{op}}$ is a strict cartesian identity-on-objects functor.*

Proof. It is clear that L_T is an identity-on-objects functor. For preservation of products we have

$$\begin{aligned}
\mathbf{Kl}(T)^{\mathrm{op}}(X, \textstyle\prod_i Y_i) &= \mathbf{Law}_n(S)(p(\textstyle\prod_i Y_i), T(X)) \\
&\cong \textstyle\prod_i \mathbf{Law}_n(S)(p(Y_i), T(X)) && (\text{Lemma A.24}) \\
&= \textstyle\prod_i \mathbf{Kl}(T)^{\mathrm{op}}(X, Y_i)
\end{aligned}$$

and this sends F to $(F \circ p(\pi_i))_i = (F^\dagger \circ L_T(\pi_i))_i$. □

Lemma A.27. *Suppose for $n \in \mathbb{N}_\omega$ that $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ is an $(n+1)^{th}$-order algebraic theory. For all $X \in \mathbb{L}_{n+1}(S)$, the algebraic theory $L /\!\!/ X$, together with $X^* : L \to L /\!\!/ X$ and the restriction of L to $\mathrm{Id} /\!\!/ X \to L /\!\!/ X$, forms the coproduct of L and $\mathrm{Id} /\!\!/ X$ in $\mathbf{Law}_{n+1}(S)$.*

Proof. Suppose that $L' : \mathbb{L}_n(S) \to \mathscr{L}'$ is an $(n+1)^{\mathrm{th}}$-order algebraic theory and that $F : L \to L'$ and $F' : \mathrm{Id} /\!\!/ X \to L'$ are morphisms in $\mathbf{Law}_{n+1}(S)$. Define $x \stackrel{\mathrm{def}}{=} F'(\pi_X) \in \mathscr{L}'(1, X)$. Consider the following diagram:

$$\begin{tikzcd}
& \mathscr{L} \arrow[dr, "X^*"] \arrow[drr, bend left, "F"] & & \\
\mathbb{L}_{n+1}(S) \arrow[ur, "L"] \arrow[rr, "L/\!\!/X"] \arrow[dr, "X^*"'] & & \mathscr{L} /\!\!/ X \arrow[r, dashed, "{[F,F']}"] & \mathscr{L}' \\
& \mathbb{L}_{n+1}(S) /\!\!/ X \arrow[ur, "L"'] \arrow[urr, bend right, "F'"'] & &
\end{tikzcd}$$

The two triangles on the left commute because L strictly preserves products. By Lemma A.21, there is a unique morphism $[F, F'] \overset{\text{def}}{=} \widehat{(F, x)} : L /\!/ X \to L'$ such that the top right commutes and $[F, F'](\pi_X) = x$. The bottom right commutes because $F' \circ X^* = L'$ (as F' is a morphism of algebraic theories), $[F, F'](L(\pi_X)) = [F, F'](\pi_X) = x$ (since L strictly preserves products), and

$$[F, F'] \circ L \circ X^* \;=\; [F, F'] \circ X^* \circ L \;=\; F \circ L \;=\; L'$$

so both F' and $[F, F'] \circ L$ are the morphism $\widehat{(L', x)} : \mathrm{Id} /\!/ X \to L'$ from Lemma A.21.

It remains to show that $[F, F']$ is unique. By the universal property of $[F, F']$, it suffices to show that if $G : L /\!/ X \to L'$ satisfies $G \circ L = F'$ then $G(\pi_X) = x$, and this is the case because $G(\pi_X) = G(L(\pi_X)) = F'(\pi_X) = x$. □

To obtain the coproduct in the form mentioned in the paper, note, for $0 < n \in \mathbb{N}_\omega$ and $Y \in \mathbb{L}_n(S)$, that $\mathrm{Id} /\!/ Y \in \mathbf{Law}_n(S)$, and $p(Y) \in \mathbf{Law}_n(S)$, where Y is viewed as an object of $\mathbb{L}_{n+1}(S)$, are equal by definition of $\lfloor - \rfloor$.

Corollary A.28. *Let $L' : \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory, for $0 < n \in \mathbb{N}_\omega$. For all $Y \in \mathbb{L}_n(S) \subseteq \mathbb{L}_{n+1}(S)$, the algebraic theory $L' /\!/ Y$, together with $Y^* : L' \to L' /\!/ Y$ and the restriction of L' to $\mathrm{Id} /\!/ Y = p(Y) \to L' /\!/ Y$ form the coproduct of L' and $p(Y)$ in $\mathbf{Law}_n(S)$.*

Proof. Immediate from Lemma A.27. □

The relative monads we construct from algebraic theories satisfy the required coproduct condition, by the following lemma.

Lemma A.29. *Let $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ be an $(n+1)^{th}$-order algebraic theory, for $0 < n \in \mathbb{N}_\omega$. For all $X \in \mathbb{L}_{n+1}(S)$ and $Y \in \mathbb{L}_n(S)$, the diagram*

$$T_L(X) \xrightarrow{T_L(\pi_X)} T_L(X \times Y) \xleftarrow{T_L(\pi_Y) \circ \eta_Y} p(Y)$$

is a coproduct in $\mathbf{Law}_n(S)$.

Proof. By Corollary A.28, the diagram

$$T_L(X) \xrightarrow{Y^*} T_L(X) /\!/ Y \leftarrow p(Y)$$

is a coproduct, where the morphism on the right sends f to $T_L(X)(f)$. This diagram is equal to the diagram in the statement of the lemma, by the definition of the simple slice. □

Lemma 6.6. *Suppose that $(T, \eta, (-)^\dagger)$ is a relative monad on $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$, where $n \in \mathbb{N}_\omega$. If $n > 0$, also assume for each $X \in \mathbb{L}_{n+1}(S)$ and $Y \in \mathbb{L}_n(S)$ that the diagram*

$$T(X) \xrightarrow{T(\pi_X)} T(X \times Y) \xleftarrow{T(\pi_Y) \circ \eta_Y} p(Y)$$

is a coproduct in $\mathbf{Law}_n(S)$. Then L_T as defined above is an $(n+1)^{th}$-order algebraic theory, and there is an isomorphism of categories T-$\mathbf{Alg} \cong L_T$-$\mathbf{TmAlg}$ commuting with the forgetful functors into $\mathbf{Law}_n(S)$. Moreover, relative monad morphisms $T \to T'$ induce morphisms $L_T \to L_{T'}$ in $\mathbf{Law}_{n+1}(S)$ functorially.

Proof. By Lemma A.26, $L_T : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ is a strict cartesian identity-on-objects functor. For exponentials, note that if $Y \in \mathbb{L}_n(S) \subseteq \mathbb{L}_{n+1}(S)$ then

$$\lceil T(X) \rceil \xrightarrow{\lceil T(\pi_X) \rceil} \lceil T(X \times Y) \rceil \xleftarrow{\lceil T(\pi_Y) \circ \eta_Y \rceil} \lceil p(Y) \rceil$$

is a coproduct in $\mathbf{Law}_{n+1}(S)$ for each $X \in \mathbb{L}_{n+1}(S)$ because $\lceil - \rceil$ is a left adjoint. Now since $\lceil T(X) \rceil /\!/ Y$ forms the coproduct of $\lceil T(X) \rceil$ and $\mathrm{Id} /\!/ Y$ (Lemma A.27) and the counit $\lceil \lfloor \mathrm{Id} /\!/ Y \rfloor \rceil \to \mathrm{Id} /\!/ Y$ is an isomorphism (Lemma A.23), we have an isomorphism $\lceil T(X \times Y)\rceil \cong \lceil T(X)\rceil /\!\!/ Y$ for each X, making the following diagram commute, where ε^{-1} is the inverse of the counit of the coreflection.

$$
\begin{tikzcd}
\mathrm{Id}/\!\!/X \arrow[rrr, "\mathrm{Id}/\!\!/\pi_X"] \arrow[dr, "\lceil\eta\rceil\circ\varepsilon^{-1}"] \arrow[dd, "Y^*"'] & & & \mathrm{Id}/\!\!/(X\times Y) \arrow[dr, "\lceil\eta\rceil\circ\varepsilon^{-1}"] \arrow[dddll, equal] & \\
& \lceil T(X)\rceil \arrow[rrr, "\lceil T(\pi_X)\rceil"] \arrow[dd, "Y^*"] & & & \lceil T(X\times Y)\rceil \arrow[dddlll, "\cong"] \\
\mathrm{Id}/\!\!/X/\!\!/Y \arrow[dr, "(\lceil\eta\rceil\circ\varepsilon^{-1})/\!\!/Y"'] & & & \mathrm{Id}/\!\!/Y \arrow[lll, "X^*/\!\!/Y"'] \arrow[uu, "\mathrm{Id}/\!\!/\pi_Y"'] \arrow[dr, "\varepsilon^{-1}"] & \\
& \lceil T(X)\rceil/\!\!/Y & & & \lceil p(Y)\rceil \arrow[lll] \arrow[uu, "\lceil T(\pi_X)\circ\eta\rceil"']
\end{tikzcd}
$$

This induces isomorphisms

$$
\begin{aligned}
L_T(X \times Y', Y) &= \mathbf{Law}_n(S)(p(Y), T(X \times Y')) \\
&\cong \lceil T(X \times Y')\rceil(1, Y) && \text{(Lemma 5.22)} \\
&\cong (\lceil T(X)\rceil /\!\!/ Y')(1, Y) && \text{(above)} \\
&\cong \lceil T(X)\rceil(1, Y^{Y'}) && \text{(exponentials in } \lceil T(X)\rceil) \\
&\cong \mathbf{Law}_n(S)(p(Y^{Y'}), T(X)) && \text{(Lemma 5.22)} \\
&= L_T(X, Y^{Y'})
\end{aligned}
$$

natural in X and Y. Now consider $X = Y^{Y'}$, and

$$\mathrm{ev}_{Y',Y} \in (\mathrm{Id}/\!\!/(X \times Y'))(1, Y) = (\mathrm{Id}/\!\!/X/\!\!/Y')(1, Y)$$

By the diagram above, $\lceil \eta_{X\times Y'}\rceil(\varepsilon^{-1}(\mathrm{ev}_{Y',Y})) \in \lceil T(X \times Y')\rceil)(1, Y)$ is sent by the isomorphism induced by the coproduct to $((\lceil\eta_X\rceil \circ \varepsilon^{-1})/\!\!/Y')(\mathrm{ev}_{Y',Y}) \in (\lceil T(X)\rceil/\!\!/Y')(1, Y)$. Using Lemma 5.22 and preservation of exponentials by $\lceil\eta\rceil$, this implies that $L_T(\mathrm{ev}_{Y',Y}) \in L_T(X \times Y', Y)$ is sent to $\eta_{Y^{Y'}}$, which is the identity on $Y^{Y'}$ in L_T, by the chain of isomorphisms above. This implies that $\mathbf{Kl}(T)^{\mathrm{op}}$ has exponentials strictly preserved by L_T, and hence that L_T is an $(n+1)^{\text{th}}$-order algebraic theory.

For the algebras, we show that T-$\mathbf{Alg}$ is the vertex of the following pullback.

$$
\begin{tikzcd}
T\text{-}\mathbf{Alg} \arrow[r, "P_T"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathbf{Kl}(T)^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "-\circ L_T"] \\
\mathbf{Law}_n(S) \arrow[r, "L' \mapsto \lceil \mathscr{L}'\rceil(1, \lceil L'\rceil(-))"'] & {[\mathbb{L}_{n+1}(S), \mathbf{Set}]}
\end{tikzcd}
$$

The unlabelled arrow is the forgetful functor. On algebras $(A : \mathbb{L}_n(S) \to \mathscr{A}, (-)^{\ddagger})$ the functor $P_T(A, (-)^{\ddagger})$ sends X to $\lceil\mathscr{A}\rceil(1, X)$ and $e \in \mathbf{Kl}(T)^{\mathrm{op}}(X, Y) = \mathbf{Law}_n(S)(p(Y), T(X))$ to

$$\lceil\mathscr{A}\rceil(1, X) \xrightarrow{\cong} \mathbf{Law}_n(S)(p(X), A) \xrightarrow{(-)^{\ddagger}} \mathbf{Law}_n(S)(T(X), A) \xrightarrow{-\circ e} \mathbf{Law}_n(S)(p(Y), A) \xrightarrow{\cong} \lceil\mathscr{A}\rceil(1, Y)$$

Preservation of identities and composition are immediate from the algebra laws. On algebra homomorphisms h, we define $P_T(h)(x) = \lceil h(x)\rceil$. The square above commutes because

$$(-)^{\ddagger} \circ L_T(e) = (-)^{\ddagger} \circ \eta \circ p(e) = - \circ p(e)$$

implies $P_T(A, (-)^{\ddagger})(L_T(e)) = \lceil A\rceil(1, \lceil A\rceil(e))$ by naturality of the universal property of p.

To show that the square is a pullback, let $F : \mathscr{C} \to \mathbf{Law}_n(S)$ and $G : \mathscr{C} \to [\mathbf{Kl}(T)^{\mathrm{op}}, \mathbf{Set}]$ be functors such that the square formed with the cospan in the diagram above commutes. We define a functor $\mathscr{C} \to T\text{-}\mathbf{Alg}$. Given $V \in \mathscr{C}$, the underlying n^{th}-order algebraic theory of the algebra is necessarily $F(V)$. Commutativity of the square implies $G(V)(X) = \mathrm{cod}(\lceil F(V) \rceil)(1, X)$, hence the hom-function of $G(V)$ defines a function

$$\lceil F(V) \rceil(1, X) \times \mathbf{Law}_n(S)(p(Y), T(X)) \;\to\; \lceil F(V) \rceil(1, Y)$$

This is natural in $Y \in \mathbb{L}_{n+1}(S)$: given $e \in \mathbf{Kl}(T)^{\mathrm{op}}(X, Y)$ and $e' \in \mathbb{L}_{n+1}(S)(Y, Z)$, we have

$$\begin{aligned} G(V)(e \circ p(e')) &= G(V)(e^{\dagger} \circ L_T(e')) && \text{(monad law)} \\ &= G(V)(L_T(e')) \circ G(V)(e) && \text{($G(V)$ preserves composition in $\mathbf{Kl}(T)^{\mathrm{op}}$)} \\ &= \mathrm{cod}(\lceil F(V) \rceil)(1, \lceil F(V) \rceil(e')) \circ G(V)(e) && \text{(square commutes)} \end{aligned}$$

Hence we have

$$\begin{aligned} & \mathbf{Law}_n(S)(p(X), F(V)) \\ \cong\ & \mathrm{cod}(\lceil F(V) \rceil)(1, X) && \text{(Lemma 5.22)} \\ \to\ & [\mathbb{L}_{n+1}(S), \mathbf{Set}](\mathbf{Law}_n(S)(p(-), T(X)), \lceil F(V) \rceil(1, -)) && \text{($G(V)$ on morphisms)} \\ \cong\ & [\mathbb{L}_{n+1}(S), \mathbf{Set}](\mathbf{Law}_n(S)(p(-), T(X)), \mathbf{Law}_n(S)(p(-), F(V))) && \text{(Lemma 5.22)} \\ \cong\ & \mathbf{Law}_n(S)(T(X), F(V)) && \text{(Lemma A.34)} \end{aligned}$$

which we use as the definition of $(-)^{\ddagger}$. This is the unique morphism such that the required diagram commutes, by density of p. The two algebra laws follow from $G(V) \circ \eta = \mathrm{id}$ and $G(V)(g^{\dagger} \circ f) = G(V)(f) \circ G(V)(g)$ because p is dense (Lemma A.34).

Hence we have defined and shown uniqueness of $\mathscr{C} \to T\text{-}\mathbf{Alg}$ on objects. This functor necessarily sends $v : V \to V'$ in $\mathscr{C}$ to $F(v) : F(V) \to F(V')$. Since the square commutes, $G(V)$ on morphisms is composition with $\lceil F(v) \rceil$, which implies that G is the postcomposition of $\mathscr{C} \to T\text{-}\mathbf{Alg}$ by P_T. It also implies, using density of p (Lemma A.34), and naturality of $G(V)$ and of the universal property of p, that $G(V)$ is an algebra homomorphism. Hence, the square above satisfies the universal property of a pullback, and we have the required isomorphisms between categories of algebras.

Given a morphism $m : T \to T'$ of relative monads, the induced morphism $L_M : \mathbf{Kl}(T)^{\mathrm{op}} \to \mathbf{Kl}(T')^{\mathrm{op}}$ is given by composition with m. This satisfies $L_M \circ L_T = L_{T'}$ because m preserves the units. We therefore have a functor from the full subcategory of $\mathbf{RMnd}(p)$ on relative monads satisfying the coproduct condition to $\mathbf{Law}_n(S)$. □

Theorem 6.7. *For $n \in \mathbb{N}_{\omega}$, the category $\mathbf{Law}_{n+1}(S)$ is equivalent to the full subcategory of $\mathbf{RMnd}(p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S))$ on relative monads $(T, \eta, (-)^{\dagger})$ such that, if $n > 0$, then, for all $X \in \mathbb{L}_{n+1}(S)$ and $Y \in \mathbb{L}_n(S)$,*

$$T(X) \xrightarrow{T(\pi_X)} T(X \times Y) \xleftarrow{T(\pi_Y) \circ \eta_Y} p(Y)$$

is a coproduct in $\mathbf{Law}_n(S)$. Moreover, there are isomorphisms between the respective categories of algebras, commuting with the forgetful functors:

$$\begin{tikzcd} T\text{-}\mathbf{Alg} \arrow[rr, "\cong"] \arrow[dr] & & L\text{-}\mathbf{TmAlg} \arrow[dl] \\ & \mathbf{Law}_n(S) & \end{tikzcd}$$

Proof. For this proof we write Ψ for the functor from $\mathbf{Law}_{n+1}(S)$ to the subcategory of relative monads from Lemma 6.5, and Φ for the functor from the subcategory of relative monads to $\mathbf{Law}_{n+1}(S)$ from Lemma 6.6. We construct an adjunction $\Psi \dashv \Phi$, with invertible unit and counit.

Given $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ in $\mathbf{Law}_{n+1}(S)$, the unit $\zeta_L : L \cong \Phi\Psi(L)$ of the adjunction is the identity-on-objects functor given on morphisms by the bijections (12),

$$\mathscr{L}(X,Y) \xrightarrow{\cong} \mathbf{Law}_n(S)(p(Y), T_L(X))$$

which are natural in the theory L. This is a morphism of $(n+1)^{\text{th}}$-order algebraic theories because it is natural in Y and the unit of T_L is given by sending the identity along these isomorphisms. This implies the inverse is also a morphism of $(n+1)^{\text{th}}$-order algebraic theories.

For the counit, suppose we have a relative monad $T \in \mathbf{RMnd}(p)$ satisfying the coproduct condition. By Lemma 6.6 and Lemma 6.5, we have isomorphisms

$$\begin{tikzcd}
T\text{-}\mathbf{Alg} \arrow[r, "\cong"] \arrow[dr] & \Phi(T)\text{-}\mathbf{TmAlg} \arrow[r, "\cong"] \arrow[d] & \Psi\Phi(T)\text{-}\mathbf{Alg} \arrow[dl] \\
& \mathbf{Law}_n(S) &
\end{tikzcd}$$

Since Alg, the semantics functor for relative monads, reflects isomorphisms (Lemma 6.4), we therefore have an isomorphism $\varpi_T : \Psi\Phi(T) \cong T$ of relative monads. This is the counit of the adjoint equivalence.

It remains to show that the triangle laws hold (which imply naturality of the counit). To show that

$$\Psi(L) \xrightarrow{\Psi(\zeta_L)} \Psi\Phi\Psi(L) \xrightarrow{\varpi_{\Psi(L)}} \Psi(L)$$

is the identity is equivalent, by Lemma 6.4, to showing that

$$\Psi(L)\text{-}\mathbf{Alg} \xrightarrow{\mathrm{Alg}\,(\varpi_{\Psi(L)})} \Psi\Phi\Psi(L)\text{-}\mathbf{Alg} \xrightarrow{\mathrm{Alg}\,(\Psi(\zeta_L))} \Psi(L)\text{-}\mathbf{Alg}$$

over $\mathbf{Law}_n(S)$ is the identity.

For each L, we have a pullback

$$\begin{tikzcd}
\Psi(L)\text{-}\mathbf{Alg} \arrow[r, "P_L"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\mathscr{L}, \mathbf{Set}]} \arrow[d, "-\circ L"] \\
\mathbf{Law}_n(S) \arrow[r, swap, "L' \mapsto \ulcorner \mathscr{L}' \urcorner(1, \ulcorner L' \urcorner(-))"] & {[\mathbb{L}_{n+1}(S), \mathbf{Set}]}
\end{tikzcd}$$

as in the proof of Lemma 6.5, so it suffices to show that

$$\Psi(L)\text{-}\mathbf{Alg} \xrightarrow{\mathrm{Alg}\,(\varpi_{\Psi(L)})} \Psi\Phi\Psi(L)\text{-}\mathbf{Alg} \xrightarrow{\mathrm{Alg}\,(\Psi(\zeta_L))} \Psi(L)\text{-}\mathbf{Alg} \xrightarrow{P_L} [\mathscr{L}, \mathbf{Set}]$$

is equal to P_L. For this we use the following diagram,

$$\begin{tikzcd}
& \Psi(L)\text{-}\mathbf{Alg} \arrow[dl, bend right, swap, "\mathrm{Alg}\,(\varpi_{\Psi(L)})"] \arrow[d, "P_{\Psi(L)}"] \arrow[dr, bend left, "P_L"] & \\
\Psi\Phi\Psi(L)\text{-}\mathbf{Alg} \arrow[r, "P_{\Phi\Psi(L)}"] \arrow[dr, bend right, swap, "\mathrm{Alg}\,(\Psi(\zeta_L))"] & {[\Phi\Psi(\mathscr{L}), \mathbf{Set}]} \arrow[r, "-\circ\zeta_L"] & {[\mathscr{L}, \mathbf{Set}]} \\
& \Psi(L)\text{-}\mathbf{Alg} \arrow[u, "P_{\Psi(L)}"] \arrow[ur, bend right, swap, "P_L"] &
\end{tikzcd} \tag{14}$$

where the functors P are defined as in the proofs of Lemmas 6.5 and 6.6. The top left follows from the definition of the counit, specifically, using the fact that the isomorphisms defining $\varpi_{\Psi(L)}$ also satisfy

$$\begin{tikzcd} \Psi(L)\text{-}\mathbf{Alg} \arrow[r, "\cong"] \arrow[dr, "P_{\Psi(L)}"'] & \Phi\Psi(L)\text{-}\mathbf{TmAlg} \arrow[r, "\cong"] \arrow[d] & \Psi\Phi\Psi(L)\text{-}\mathbf{Alg} \arrow[dl, "P_{\Psi\Phi(L)}"] \\ & {[\Phi\Psi(\mathscr{L}), \mathbf{Set}]} & \end{tikzcd}$$

where the unlabelled functor is from the definition of term algebra. The two triangles on the right of (14) commute by the definitions of P_L and $P_{\Psi(L)}$. The only non-trivial part is

$$\begin{aligned} P_{\Phi(L)}(A, (-)^{\ddagger})(\zeta_L(e)) &= P_{\Phi(L)}(A, (-)^{\ddagger})(\lfloor L/\!\!/e \rfloor \circ \eta) && \text{(naturality of (12), definition of } \eta) \\ &= P_L(e) && \text{(definitions of } P_L \text{ and } P_{\Phi(L)}) \end{aligned}$$

where η is the unit of the relative monad $\Psi(L)$. For the bottom left of (14), the only non-trivial part is showing

$$P_{\Psi(L)}(\mathrm{Alg}(\Psi(\zeta_L))\,(A, (-)^{\ddagger})(e)) = P_{\Phi\Psi(L)}(A, (-)^{\ddagger})(e)$$

for $\Psi\Phi\Psi(L)$-algebras $(A, (-)^{\ddagger})$. To do this, note that the induced $\Psi(L)$-algebra has $(-)^{\ddagger} \circ \Psi(\zeta_L)$ as its extension operator. This case then follows from the definitions of $P_{\Psi(L)}$ and $P_{\Phi\Psi(L)}$ and the fact that the unit of the relative monad $\Psi\Phi\Psi(L)$ factors through $T_L(Y)$:

$$p(Y) \xrightarrow{\eta_Y} T_L Y \longrightarrow T_{\Phi\Psi L} Y$$

For the other triangle law

$$\Phi(T) \xrightarrow{\zeta_{\Phi(T)}} \Phi\Psi\Phi(T) \xrightarrow{\Phi(\varpi_T)} \Phi(T)$$

we calculate ϖ_T. Consider the free T-algebra $(T(X), (-)^{\dagger})$. From their definitions, we determine that the isomorphisms defining ϖ_T send the free algebra to a $\Psi\Phi(T)$-algebra $(T(X), (-)^{\ddagger})$, with $(-)^{\ddagger}$ given by

$$\begin{aligned} & \mathbf{Law}_n(S)(p(Y), T(X)) \\ \xrightarrow{(-)^{\dagger}}\ & \mathbf{Law}_n(S)(T(Y), T(X)) \\ \cong\ & [\mathbb{L}_{n+1}(S)^{\mathrm{op}}, \mathbf{Set}](\mathbf{Kl}(T)^{\mathrm{op}}(Y, -), \mathbf{Law}_n(S)(p(-), T(X))) && \text{(Lemma A.34)} \\ \cong\ & [\mathbb{L}_{n+1}(S)^{\mathrm{op}}, \mathbf{Set}](\mathbf{Kl}(\Psi\Phi(T))^{\mathrm{op}}(Y, -), \mathbf{Law}_n(S)(p(-), T(X))) && \text{((12))} \\ \cong\ & \mathbf{Law}_n(S)(\lfloor \Phi(T)/\!\!/Y \rfloor, T(X)) && \text{(Lemma A.34)} \end{aligned}$$

On morphisms $F \in \mathbf{Law}_n(S)(p(Z), \Psi\Phi T(Y)) = \mathbf{Kl}(\Psi\Phi(T))^{\mathrm{op}}(Y, Z)$, the map $\Phi(\varpi_T)$ is given by

$$\Phi(\varpi_T)(F) \;=\; \varpi_T \circ F \;=\; {\eta_Y}^{\ddagger} \circ F \;\in\; \mathbf{Kl}(T)^{\mathrm{op}}(Y, Z)$$

which is therefore just the action of sending F along the bijections (12). By definition, this is the inverse of the action of $\zeta_{\Phi(T)}$ on morphisms, proving the triangle law. □

A.7 Local strong presentability

Lemma A.30. *A cocomplete category $\mathscr{C}$ is locally strongly finitely presentable exactly when there exists a small category $\mathscr{C}_{\mathrm{sf}}$ and locally strongly finitely presentable (Definition 7.4) functor $p : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$.*

Proof. If $\mathscr{C}$ is strongly finitely presentable, we obtain such a functor by Adámek and Rosický [2001, Lemma 3.8]. On the other hand, given p, density implies that each object $X \in \mathscr{C}$ is a canonical colimit and, since $p \downarrow X$ is sifted, this colimit is sifted. Hence we can take the image of p as the required set of strongly finitely presentable objects. □

We show that each $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ is locally strongly finitely presentable. Limits and (sifted) colimits in the functor category $\mathbf{Law}_0(S) = \mathbf{Set}^S$ are computed pointwise; we show that for $n > 0$ they are given as described in Section 7.

Lemma A.31. *Let $D : \mathbb{I} \to \mathbf{Law}_n(S)$ be a diagram, where $\mathbb{I}$ is sifted and $n > 0$, and denote by L_i the theory $D(i)$. Define $L : \mathbb{L}_n(S) \to \mathscr{L}$ as in Section 7, with $\mathscr{L}(X, Y) = \mathrm{colim}_{i\in\mathbb{I}}(\mathscr{L}_i(X, Y))$. Then $L : \mathbb{L}_n(S) \to \mathscr{L}$ forms an n^{th}-order algebraic theory, and the family of functors $(\iota_i : \mathscr{L}_i \to \mathscr{L})_{i\in\mathbb{I}}$, given by coprojections on each hom-set, is a colimiting cocone in $\mathbf{Law}_n(S)$.*

Proof. It is trivial to show that $\mathscr{L}$ is a category and that ι_i and L are identity-on-objects functors. The category $\mathscr{L}$ has finite products because

$$\begin{aligned} \mathrm{colim}_{i\in\mathbb{I}}(\mathscr{L}_i(X, \textstyle\prod_j Y_j)) &\cong \mathrm{colim}_{i\in\mathbb{I}}(\textstyle\prod_j \mathscr{L}_i(X, Y_j)) && \text{(products in } \mathscr{L}_i) \\ &\cong \textstyle\prod_j \mathrm{colim}_i(\mathscr{L}_i(X, Y_j)) && (\mathbb{I} \text{ is sifted}) \end{aligned}$$

and L strictly preserves them as it sends the identity on $\prod_j X_j$ to $(L(\pi_j))_j$. Exponentials in $\mathscr{L}$ are given by

$$\mathrm{colim}_{i\in\mathbb{I}}(\mathscr{L}_i(X \times Y, Z)) \cong \mathrm{colim}_{i\in\mathbb{I}}(\mathscr{L}_i(X, Z^Y)) \qquad \text{(exponentials in } \mathscr{L}_i)$$

and this sends $\mathrm{ev}_{Y,Z}$ to the identity on Z^Y.

Hence L is an n^{th}-order algebraic theory, and the functors $\iota_i : \mathscr{L}_i \to \mathscr{L}$ form a cocone in $\mathbf{Law}_n(S)$. To show this cocone is colimiting, suppose that $L' : \mathbb{L}_n(S) \to \mathscr{L}'$ is an n^{th}-order algebraic theory, and that the functors $F_i : \mathscr{L}_i \to \mathscr{L}'$ form a cocone. The functions $[F_i]_{i\in\mathbb{I}} : \mathscr{L}(X, Y) \to \mathscr{L}'(X, Y)$ form an identity-on-objects functor $\mathscr{L} \to \mathscr{L}'$. Is is easy to see that this is the unique universal morphism in $\mathbf{Law}_n(S)$. □

Lemma A.32. *$\mathbf{Law}_n(S)$ is complete for all $n > 0$.*

Proof. Limits of diagrams $L : \mathbb{I} \to \mathbf{Law}_n(S)$ are constructed as in Lemma A.31, except with lim instead of colim and projections instead of coprojections. The proof is then similar, using the fact that small limits in $\mathbf{Set}$ commute with products. □

Next, we show the required properties of p.

Lemma A.33. *The functor $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ is fully faithful for all $n \in \mathbb{N}_\omega$.*

Proof. Let $X, Y \in \mathbb{L}_n(S)^{\mathrm{op}}$. The action of p on morphisms is given by

$$\begin{aligned} \mathbb{L}_{n+1}(S)^{\mathrm{op}}(X, Y) &= \mathbb{L}_{n+1}(S)(Y, X) \\ &= (\mathrm{Id}/\!/Y)(1, X) && (\text{definition of } \mathrm{Id}/\!/Y) \\ &\cong \mathrm{cod}(\lceil p(Y) \rceil)(1, X) && (\text{Lemma A.23}) \\ &\cong \mathbf{Law}_n(S)(p(X), p(Y)) && (\text{Lemma 5.22}) \end{aligned}$$

□

Lemma A.34. *The functor $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ is dense for all $n \in \mathbb{N}_\omega$.*

Proof. Lemma 5.22 implies the nerve functor N_p is naturally isomorphic to $L \mapsto \lceil \mathscr{L} \rceil(1, \lceil L \rceil(-))$, so it suffices that the latter is fully faithful (Lemma A.12). □

Lemma A.35. *The comma category $p \downarrow L$ is sifted for each $n \in \mathbb{N}_\omega$ and $L \in \mathbf{Law}_n(S)$.*

Proof. Since $\mathbb{L}_{n+1}(S)^{\mathrm{op}}$ has finite coproducts and these are preserved by $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ (Lemma A.24), the comma category $p \downarrow L$ has finite coproducts. It is therefore sifted by Adámek and Rosický [2001, Remark 1.2(d)]. □

Finally, we show that the image of p contains only strongly finitely presentable objects.

Lemma A.36. *The n^{th}-order algebraic theory $p(X)$ is strongly finitely presentable for each $n \in \mathbb{N}_\omega$ and $X \in \mathbb{L}_{n+1}(S)$.*

Proof. For each diagram $L : \mathbb{I} \to \mathbf{Law}_n(S)$ with $\mathbb{I}$ sifted we have

$$\begin{aligned}
\mathrm{colim}_{i\in\mathbb{I}}(\mathbf{Law}_n(S)(p(X), L(i))) &\cong \mathrm{colim}_{i\in\mathbb{I}}(\mathrm{cod}(\lceil L(i) \rceil)(1, X)) && \text{(Lemma 5.22)} \\
&= (\mathrm{colim}_{i\in\mathbb{I}}\, \mathrm{cod}(\lceil L(i) \rceil))(1, X) \\
& && \text{(construction of sifted colimits in } \mathbf{Law}_{n+1}(S)) \\
&\cong \mathrm{cod}(\lceil \mathrm{colim}\, L \rceil)(1, X) && \text{(left adjoints preserve colimits)} \\
&\cong \mathbf{Law}_n(S)(p(X), \mathrm{colim}\, L) && \text{(Lemma 5.22)}
\end{aligned}$$

and the composition of these isomorphisms is the canonical function. □

Theorem 7.5. *For all $n \in \mathbb{N}_\omega$ the category $\mathbf{Law}_n(S)$ is cocomplete, and the functor $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ is locally strongly finitely presentable. Hence $\mathbf{Law}_n(S)$ is locally strongly finitely presentable.*

Proof. The category $\mathbf{Law}_n(S)$ has sifted colimits (Lemma A.31), and $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ provides a small, full (Lemma A.33), dense (Lemma A.34) subcategory of strongly finitely presentable objects (Lemma A.36). Furthermore, the comma category $p \downarrow L$ is sifted for every $L \in \mathbf{Law}_n(S)$ (Lemma A.35). Hence it is a *generalised variety* in the sense of Adámek and Rosický [2001]. Completeness (Lemma A.32) therefore implies cocompleteness by [Adámek and Rosický 2001, Remark 4.8], and hence that $p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S)$ and $\mathbf{Law}_n(S)$ are locally strongly finitely presentable. □

Theorem 7.6 (Bicompleteness of term algebras and strict models). *Let $L : \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. L-$\mathbf{TmAlg}$ is locally strongly finitely presentable, and in particular complete and cocomplete. $L/\mathbf{Law}_n(S)$ is therefore also complete and cocomplete.*

Proof. Follows immediately from Proposition 5.15 and the fact that $\mathbf{Cart}(\mathscr{L}, \mathbf{Set}) \simeq \mathbf{Sind}(\mathscr{L}^{\mathrm{op}})$, the sifted cocompletion of $\mathscr{L}^{\mathrm{op}}$, by Corollary 2.8 of Adámek and Rosický [2001]. Note that this also implies that every theory is equivalent to the opposite of the subcategory of strongly finitely presentable objects of its category of term algebras. The category of strict models is a coreflective subcategory of the category of term algebras (Proposition 5.20), and so inherits limits and colimits from L-$\mathbf{TmAlg}$ [Riehl 2017, Proposition 4.5.15]. □

A.8 Monad–theory correspondence

The following lemma is essentially Corollary 2.7 of Adámek, Milius, Sousa, and Wissmann [2019].

Lemma A.37. *Suppose that $p : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$ is locally strongly finitely presentable. A functor $F : \mathscr{C} \to \mathscr{D}$ preserves sifted colimits iff the functions that send morphisms $f \in \mathscr{D}(F(X), X')$ to natural transformations $f \circ F(-) : \mathscr{C}(p(-), X) \Rightarrow \mathscr{D}(F(p(-)), X')$ are bijections.*

Proof. Let $D : \mathbb{I} \to \mathscr{C}$ be a sifted diagram. The functor F preserves sifted colimits iff the function sending each morphism $f \in \mathscr{D}(F(\mathrm{colim}\, D), X)$ to the natural transformation $f \circ F(\iota_{i\in\mathbb{I}}) : F(D_{i\in\mathbb{I}}) \to X$ is invertible for each sifted diagram D.

Suppose that the function in the statement of the lemma is a bijection. Then

$$\begin{aligned}
\mathscr{D}(F(\mathrm{colim}\, D), X) &\cong [\mathscr{C}_{\mathrm{sf}}^{\mathrm{op}}, \mathbf{Set}](\mathscr{C}(p(-), \mathrm{colim}\, D), \mathscr{D}(F(p(-)), X)) && \text{(bijection)} \\
&\cong [\mathscr{C}_{\mathrm{sf}}^{\mathrm{op}}, \mathbf{Set}](\mathrm{colim}_i(\mathscr{C}(p(-), D_i)), \mathscr{D}(F(p(-)), X)) \\
& && (p \text{ is locally strongly finitely presentable, } D \text{ is sifted}) \\
&\cong [\mathbb{I}, [\mathscr{C}_{\mathrm{sf}}^{\mathrm{op}}, \mathbf{Set}]](\mathscr{C}(p(=), D_{(-)}), \mathscr{D}(F(p(=)), X)) && \text{(colimits)} \\
&\cong [\mathbb{I}, \mathscr{D}](F \circ D, X) && \text{(bijection)}
\end{aligned}$$

and this sends f to $f \circ F(\iota_i)$. Conversely, if F preserves sifted colimits then

$$\begin{aligned}
\mathscr{D}(F(X), X') &\cong \mathscr{D}(F(\mathrm{colim}(p \downarrow X \to \mathscr{C}_{\mathrm{sf}} \to \mathscr{C})), X') \\
&\qquad\qquad (p \text{ is locally strongly finitely presentable}) \\
&\cong [p \downarrow X, \mathscr{D}](F(p(\mathrm{dom}\,(-))), X') \qquad (F \text{ preserves sifted colimits}) \\
&\cong [\mathscr{C}_{\mathrm{sf}}^{\mathrm{op}}, \mathbf{Set}](\mathscr{C}(p(-), X), \mathscr{D}(F(p(-)), X')) \qquad (\text{comma categories})
\end{aligned}$$

and this sends f to $f \circ F(-)$. □

We show that for any locally strongly finitely presentable $p : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$ there is an equivalence $[\mathscr{C}_{\mathrm{sf}}, \mathscr{C}] \simeq [\mathscr{C}, \mathscr{C}]_{\mathrm{sf}}$, where $[\mathscr{C}, \mathscr{C}]_{\mathrm{sf}} \hookrightarrow [\mathscr{C}, \mathscr{C}]$ is the full subcategory on the sifted-cocontinuous functors. To do so, we need the universal property of a pointwise left Kan extension.

Definition A.38. A pair $(\mathrm{Lan}_p F, \lambda)$ of a functor $\mathrm{Lan}_p F : \mathscr{C} \to \mathscr{D}$ and natural transformation $\lambda : F \Rightarrow \mathrm{Lan}_p F \circ p$ is the *pointwise left Kan extension* of $F : \mathscr{C}_{\mathrm{sf}} \to \mathscr{D}$ along $p : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$ if, for each natural transformation $\sigma : \mathscr{C}(p(-), X) \Rightarrow \mathscr{D}(F(-), Y)$, there is a unique morphism $[\sigma] : \mathrm{Lan}_p F(X) \to Y$ such that, for all $\alpha : p(X') \to X$, the following diagram commutes.

$$\begin{tikzcd}
F(X') \arrow[r, "\lambda_{X'}"] \arrow[d, "\sigma_{X'}(\alpha)"'] & \mathrm{Lan}_p F(p(X')) \arrow[d, "\mathrm{Lan}_p F(\alpha)"] \\
Y & \mathrm{Lan}_p F(X) \arrow[l, "{[\sigma]}"]
\end{tikzcd}$$

This definition is equivalent to asking each $(\mathrm{Lan}_p F(X), \lambda_X)$ to be the colimit of $p \downarrow X \to \mathscr{C} \xrightarrow{F} \mathscr{D}$, which is the usual formula for the left Kan extension as an (ordinary) colimit (see, for example, Borceux [1994a, Theorem 3.7.2]).

Pointwise left Kan extensions along p exist for any locally strongly finitely presentable p: the natural transformation $\lambda_X : F(X) \to \mathrm{Lan}_p F(p(X))$ is given by the coprojection of the identity on $p(X)$; and the action of $\mathrm{Lan}_p F$ on a morphism $f : X \to Y$ is given by composing the diagram $p \downarrow Y \to \mathscr{C}_{\mathrm{sf}} \xrightarrow{F} \mathscr{D}$ with the functor $p \downarrow f : p \downarrow X \to p \downarrow Y$.

Pointwise Kan extensions are in particular Kan extensions: given any functor $G : \mathscr{C} \to \mathscr{D}$ and natural transformation $\sigma : F \Rightarrow G \circ p$, the universal arrow $\widetilde{\sigma} : \mathrm{Lan}_p F \Rightarrow G$ is given by $\widetilde{\sigma}_X \stackrel{\mathrm{def}}{=} [\alpha \mapsto G(\alpha) \circ \sigma]$.

Lemma A.39. *Suppose that $p : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$ is locally strongly finitely presentable, and that $\mathscr{D}$ has sifted colimits. Pointwise left Kan extensions form an adjunction $\mathrm{Lan}_p \dashv (- \circ p)$. Moreover, the image of Lan_p contains only sifted-cocontinuous functors, and hence the adjunction restricts to an adjoint equivalence $[\mathscr{C}_{\mathrm{sf}}, \mathscr{D}] \simeq [\mathscr{C}, \mathscr{D}]_{\mathrm{sf}}$.*

Proof. We note that the equivalence follows abstractly from Adámek, Borceux, Lack, and Rosický [2002, Theorem 5.5], but we will give a more direct proof.

It is well-known that pointwise left Kan extensions form an adjunction [Mac Lane 1978, Chapter X]. We show that $\mathrm{Lan}_p F$ is sifted-cocontinuous. Since p is fully faithful, $\lambda : F \Rightarrow \mathrm{Lan}_p F \circ p$ is a natural isomorphism: the inverse of λ_X is the unique $\lambda_X^{-1} : \mathrm{Lan}_p F(p(X)) \to F(X)$ corresponding to the natural transformation

$$\mathscr{C}(p(-), p(X)) \xrightarrow{\cong} \mathscr{C}_{\mathrm{sf}}(-, X) \xrightarrow{F} \mathscr{D}(F(-), F(X))$$

Hence,

$$\begin{aligned}
\mathscr{D}(\mathrm{Lan}_p F(X), Y) &\cong [\mathscr{C}_{\mathrm{sf}}^{\mathrm{op}}, \mathbf{Set}](\mathscr{C}(p(-), X), \mathscr{D}(F(-), Y)) && \text{(pointwise Kan extension)} \\
&\cong [\mathscr{C}_{\mathrm{sf}}^{\mathrm{op}}, \mathbf{Set}](\mathscr{C}(p(-), X), \mathscr{D}(\mathrm{Lan}_p F(p(-)), Y)) && \text{(λ is invertible)}
\end{aligned}$$

and this sends $f \in \mathscr{D}(\mathrm{Lan}_p F(X), Y)$ to $f \circ \mathrm{Lan}_p F(-)$, so $\mathrm{Lan}_p F$ is sifted-cocontinuous by Lemma A.37.

Finally, we show that the restriction of the adjunction to sifted-cocontinuous functors is an adjoint equivalence. For this it suffices to show that if $G : \mathscr{C} \to \mathscr{D}$ is sifted-cocontinuous then the counit $\varepsilon_X \overset{\mathrm{def}}{=} \widetilde{\mathrm{id}_{G \circ p}} : \mathrm{Lan}_p(G \circ p) \Rightarrow G$ on G is invertible. The inverse $\varepsilon_X^{-1} : G(X) \to \mathrm{Lan}_p(G \circ p)(X)$ is the unique morphism such that the natural transformation

$$\mathscr{C}(p(-), X) \xrightarrow{\alpha \mapsto \mathrm{Lan}_p(G \circ p)(\alpha) \circ \lambda_X} \mathscr{D}(G(p(-)), \mathrm{Lan}_p(G \circ p)(X))$$

is equal to $\varepsilon_X^{-1} \circ G(-)$; this exists by Lemma A.37 because G is sifted-cocontinuous. To show $\varepsilon_X \circ \varepsilon_X^{-1} = \mathrm{id}_{G(X)}$, it suffices, by Lemma A.37, to show that, for each $\alpha : p(X') \to X$, we have $\varepsilon_X \circ \varepsilon_X^{-1} \circ G(\alpha) = G(\alpha)$. This is the case because

$$\begin{aligned}
\varepsilon_X \circ \varepsilon_X^{-1} \circ G(\alpha) &= \varepsilon_X \circ \mathrm{Lan}_p(G \circ p)(\alpha) \circ \lambda_A && \text{(definition of ε^{-1})} \\
&= G(\alpha) \circ \varepsilon_{p(A)} \circ \lambda_A && \text{(naturality of ε)} \\
&= G(\alpha) && \text{(definition of ε)}
\end{aligned}$$

Finally, to show $\varepsilon_X^{-1} \circ \varepsilon_X = \mathrm{id}_{\mathrm{Lan}_p(G \circ p) X}$, given any $\alpha : p(X') \to X$ we have

$$\begin{aligned}
\varepsilon_X^{-1} \circ \varepsilon_X \circ \mathrm{Lan}_p(G \circ p)(\alpha) \circ \lambda_{X'} &= \varepsilon_X^{-1} \circ G(\alpha) && \text{(definition of ε)} \\
&= \mathrm{Lan}_p(G \circ p)(\alpha) \circ \lambda_{X'} && \text{(definition of ε^{-1})}
\end{aligned}$$

which suffices by the definition of pointwise left Kan extension. □

Lemma A.40. *If $p : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$ is strongly finitely accessible and (T, η, μ) is a sifted-cocontinuous monad on $\mathscr{C}$, then then the functor T-$\mathbf{Alg} \to (T \circ p)$-$\mathbf{Alg}$ defined in Section 8 is an isomorphism.*

Proof. Since T preserves sifted colimits, Lemma A.37 induces a unique morphism $a : T(A) \to A$ such that $a \circ T(-) = (-)^{\ddagger}$. Then (A, a) is a T-algebra because p is dense and, given any $\alpha : p(X) \to A$, we have

$$\begin{aligned}
a \circ \eta_A \circ \alpha &= a \circ T(\alpha) \circ \eta_{p(X)} && \text{(naturality of η)} \\
&= \alpha^{\ddagger} \circ \eta_{p(X)} && \text{(definition of a)} \\
&= \alpha && \text{(algebra)}
\end{aligned}$$

and, given any $\alpha' : p(X') \to T(p(X))$,

$$\begin{aligned}
a \circ T(a) \circ T(T(\alpha)) \circ T(\alpha') &= a \circ T(\alpha^{\ddagger}) \circ T(\alpha') && \text{(definition of a)} \\
&= (\alpha^{\ddagger} \circ \alpha')^{\ddagger} && \text{(definition of a)} \\
&= \alpha^{\ddagger} \circ \alpha'^{\dagger} && \text{(algebra)} \\
&= a \circ T(\alpha) \circ \mu \circ T(\alpha') && \text{(definitions of a and $(-)^{\dagger}$)} \\
&= a \circ \mu \circ T(T(\alpha)) \circ T(\alpha') && \text{(naturality of μ)}
\end{aligned}$$

If $h : (A, (-)^{\ddagger}) \to (A', (-)^{\ddagger})$ is a $(T \circ p)$-algebra homomorphism, then it is a T-algebra homomorphism, because T is sifted-cocontinuous, and, for all $\alpha : p(X) \to A$,

$$\begin{aligned} a' \circ T(h) \circ T(\alpha) &= (h \circ \alpha)^{\ddagger} && \text{(definition of } a'\text{)} \\ &= h \circ \alpha^{\ddagger} && \text{(algebra homomorphism)} \\ &= h \circ a \circ T(\alpha) && \text{(definition of } a\text{)} \end{aligned}$$

Hence we have a functor $(T \circ p)$-**Alg** $\to T$-**Alg**. Preservation of sifted colimits immediately implies that it is the inverse of T-**Alg** $\to (T \circ p)$-**Alg**. □

Theorem 8.1. *Suppose that $p : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$ is locally strongly finitely presentable. The construction above forms an adjunction (on the left), which restricts to an equivalence of categories on the sifted-cocontinuous monads (on the right).*

$$\mathbf{RMnd}(p) \underset{-\circ p}{\overset{\mathrm{Lan}_p}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Mnd}(\mathscr{C}) \qquad \mathbf{RMnd}(p) \underset{-\circ p}{\overset{\mathrm{Lan}_p}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathbf{Mnd}_{\mathrm{sf}}(\mathscr{C})$$

Moreover, there are isomorphisms between the corresponding categories of algebras, and these commute with the forgetful functors, as below, for all $T \in \mathbf{Mnd}_{\mathrm{sf}}(\mathscr{C})$ and $T' \in \mathbf{RMnd}(p)$.

$$\begin{tikzcd} T\text{-}\mathbf{Alg} \arrow[rr, "\cong"] \arrow[dr] & & (T \circ p)\text{-}\mathbf{Alg} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd} \qquad \begin{tikzcd} (\mathrm{Lan}_p T')\text{-}\mathbf{Alg} \arrow[rr, "\cong"] \arrow[dr] & & T'\text{-}\mathbf{Alg} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

Proof. The *well-behavedness* conditions given by Altenkirch, Chapman, and Uustalu [2010, Definition 4.1] all hold for p. In particular, the condition that the canonical morphism

$$\mathrm{Lan}_p(\mathscr{C}(p(X), F(-)))(Y) \to \mathscr{C}(p(X), \mathrm{Lan}_p F(Y))$$

is invertible, for all $F : \mathscr{C}_{\mathrm{sf}} \to \mathscr{C}$ and $X, Y \in \mathscr{C}_{\mathrm{sf}}$, holds because $\mathscr{C}(p(X), -)$ preserves sifted colimits, and hence pointwise Kan extensions along p; Theorem 4.8 ibid. then gives us the adjunction in the theorem statement. Lemma A.39 implies this adjunction restricts to an equivalence, since (relative) monad morphisms are closed under taking inverses.

For the algebras, the triangle on the left above comes from Lemma A.40. The other is then

$$(\mathrm{Lan}_p T)\text{-}\mathbf{Alg} \;\cong\; (\mathrm{Lan}_p T \circ p)\text{-}\mathbf{Alg} \;\cong\; T\text{-}\mathbf{Alg}$$

using $(\mathrm{Lan}_p T) \circ p \cong T$. □

Theorem 8.2. *For $n \in \mathbb{N}_{\omega}$, the following are equivalent.*

(1) *The category $\mathbf{Law}_{n+1}(S)$ of $(n+1)^{th}$-order algebraic theories.*
(2) *The full subcategory of $\mathbf{RMnd}(p : \mathbb{L}_{n+1}(S)^{\mathrm{op}} \to \mathbf{Law}_n(S))$ on relative monads $(T, \eta, (-)^{\dagger})$ such that, if $n > 0$, then, for all $X \in \mathbb{L}_{n+1}(S)$ and $Y \in \mathbb{L}_n(S)$,*

$$T(X) \xrightarrow{T(\pi_X)} T(X \times Y) \xleftarrow{T(\pi_Y) \circ \eta_Y} p(Y)$$

is a coproduct in $\mathbf{Law}_n(S)$.
(3) *The full subcategory of $\mathbf{Mnd}_{\mathrm{sf}}(\mathbf{Law}_n(S))$ on monads (T, η, μ) such that, if $n > 0$, then, for all $L \in \mathbf{Law}_n(S)$ and $Y \in \mathbb{L}_n(S)$,*

$$T(L) \xrightarrow{T(\varkappa_L)} T(L + p(Y)) \xleftarrow{T(\varkappa_{p(Y)}) \circ \eta_{p(Y)}} p(Y)$$

is a coproduct in $\mathbf{Law}_n(S)$.

Moreover, if an $(n+1)^{th}$-order algebraic theory L, relative monad T, and monad $\hat{T}$ are related by these equivalences, then there are isomorphisms between the respective categories of categories of algebras commuting with the forgetful functors:

$$\begin{tikzcd} \hat{T}\text{-}\mathbf{Alg} \arrow[r, "\cong"] \arrow[dr] & T\text{-}\mathbf{Alg} \arrow[r, "\cong"] \arrow[d] & L\text{-}\mathbf{TmAlg} \arrow[dl] \\ & \mathbf{Law}_n(S) & \end{tikzcd}$$

Proof. The equivalence between (1) and (2) is covered in Theorem 6.7.

For (2) $\simeq$ (3), it suffices to show that the equivalence in Theorem 8.1 restricts to one involving the coproduct conditions. Let T be a monad as in (3). Since p sends products in $\mathbb{L}_{n+1}(S)$ to coproducts in $\mathbf{Law}_n(S)$ (Lemma A.24), we have

$$\begin{tikzcd} T(p(X)) \arrow[r, "T\iota_{p(X)}"] \arrow[dr, "T(p(\pi_X))"'] & T(p(X)+p(Y)) \arrow[d, phantom, "\cong"] & p(Y) \arrow[l, "T\iota_{p(Y)}\circ\eta_{p(Y)}"'] \arrow[dl, "T(p(\pi_X))\circ\eta_{p(Y)}"] \\ & T(p(X\times Y)) & \end{tikzcd}$$

The top is a coproduct diagram, hence so is the bottom, and the induced relative monad $T \circ p$ is as in (2). In the other direction, since each monad T as in (3) preserves sifted colimits, colimits commute, and each $L \in \mathbf{Law}_n(S)$ is a canonical sifted colimit (Theorem 7.5), the canonical morphism

$$\mathrm{colim}_{(\alpha: p(X)\to L)\in p\downarrow L}(T(p(X)+p(Y))) \;\to\; T(L+p(Y))$$

is invertible. The canonical morphism $p(X)+p(Y) \to p(X\times Y)$ is also invertible (Lemma A.24), so

$$\mathrm{colim}_{(\alpha: p(X)\to L)\in p\downarrow L}(T(p(X\times Y))) \;\cong\; T(L+p(Y))$$

If T came from a relative monad as in (2), then, up to natural isomorphism, the relative monad is given by $T \circ p$, and each $T(p(X\times Y))$ is a coproduct. Hence $T(L+p(Y))$ forms the coproduct as required.

The isomorphisms between categories of algebras follow from Theorem 8.1 and Lemma 6.6. □

Proposition A.41. *Suppose that $L : \mathbb{L}_{n+1}(S) \to \mathscr{L}$ is a $(n+1)^{th}$-order algebraic theory, where $n \in \mathbb{N}_\omega$. The monad induced by the monad correspondence is isomorphic to the monad induced by the adjunction*

$$\begin{tikzcd} \mathbf{Law}_n(S) \arrow[r, shift left, "\lceil - \rceil"] \arrow[r, phantom, "\perp"] & \mathbf{Law}_{n+1}(S) \arrow[l, shift left, "\lfloor - \rfloor"] \arrow[r, shift left, "L+(-)"] \arrow[r, phantom, "\perp"] & L/\mathbf{Law}_{n+1}(S) \arrow[l, shift left, "U"] \end{tikzcd}$$

Proof. The monad $\lfloor L + \lceil - \rceil \rfloor$ induced by the adjunction is sifted-cocontinuous, because each of the functors $\lceil - \rceil$, $L + (-)$ and $\lfloor - \rfloor$ are (the latter due to the explicit construction of sifted colimits in $\mathbf{Law}_{n+1}(S)$). By Theorem 8.1, it therefore suffices to show that the induced relative monad $\lfloor L + \lceil p(-) \rceil \rfloor$ is the same as the relative monad T_L constructed from L.

For each $X \in \mathbb{L}_{n+1}(S)^{\mathrm{op}}$ we have an isomorphism

$$\lfloor L + \lceil p(X) \rceil \rfloor \;\xrightarrow{\cong}\; \lfloor L + (\mathrm{Id}/\!\!/X) \rfloor \;\xrightarrow{\cong}\; \lfloor L/\!\!/X \rfloor \;=\; T_L(X)$$

by Lemmas A.23 and A.27. It suffices to show this forms a morphism of relative monads. Preservation of the unit follows from one of the triangle laws of the coreflection. For preservation of the Kleisli extension, given $F : p(X) \to \lfloor L + \lceil p(Y) \rceil \rfloor$, we have to show that the two morphisms

$$H : \lfloor L + \lceil p(X) \rceil \rfloor \to \lfloor L /\!\!/ Y \rfloor$$

obtained using the Kleisli extensions of the two relative monads, are equal. To do this, note that by the adjunction

$$\begin{tikzcd} \mathbf{Law}_n(S) \arrow[r, shift left, "\lceil - \rceil"] \arrow[r, phantom, "\perp"] & \mathbf{Law}_{n+1}(S) \arrow[l, shift left, "\lfloor - \rfloor"] \arrow[r, shift left, "L+(-)"] \arrow[r, phantom, "\perp"] & L/\mathbf{Law}_{n+1}(S) \arrow[l, shift left, "U"] \end{tikzcd}$$

there is exactly one $G : L + \lceil p(X) \rceil \to L /\!\!/ Y$ in $L/\mathbf{Law}_{n+1}(S)$ such that the following square commutes, where η is the unit of the adjunction.

$$\begin{tikzcd} p(X) \arrow[r, "\eta_{p(X)}"] \arrow[d, "F"'] & \lfloor L + \lceil p(X) \rceil \rfloor \arrow[d, "\lfloor G \rfloor"] \\ \lfloor L + \lceil p(Y) \rceil \rfloor \arrow[r, "\cong"'] & \lfloor L /\!\!/ Y \rfloor \end{tikzcd}$$

Let $G' : L + \lceil p(X) \rceil \to L + \lceil p(Y) \rceil$ be the unique morphism of $L/\mathbf{Law}_{n+1}(S)$ such that the following triangle commutes.

$$\begin{tikzcd} p(X) \arrow[r, "\eta_{p(X)}"] \arrow[dr, "F"'] & \lfloor L + \lceil p(X) \rceil \rfloor \arrow[d, "\lfloor G' \rfloor"] \\ & \lfloor L + \lceil p(Y) \rceil \rfloor \end{tikzcd}$$

The image of the composite $L + \lceil p(X) \rceil \xrightarrow{\lfloor G' \rfloor} L + \lceil p(Y) \rceil \xrightarrow{\lfloor \cong \rfloor} L /\!\!/ Y$ under $\lfloor - \rfloor$ makes the preceding square commute, so the composite is a candidate for G, and thus $\lfloor G \rfloor$ is by definition the H induced by the relative monad constructed from the coslice adjunction.

On the other hand, the morphism

$$p(X) \xrightarrow{F} \lfloor L + \lceil p(Y) \rceil \rfloor \xrightarrow{\lfloor \cong \rfloor} \lfloor L /\!\!/ Y \rfloor$$

is sent by the bijections (12) to some morphism $f \in \mathscr{L}(Y, X)$. Then

$$L + \lceil p(X) \rceil \xrightarrow{\cong} L /\!\!/ X \xrightarrow{L /\!\!/ f} L /\!\!/ Y$$

is also a candidate for G, so is necessarily equal to the previous candidate. In this case, $\lfloor G \rfloor$ is the H induced by the relative monad T_L, and so the two morphisms H are equal. □

A.9 Zeroth-order algebraic theories

0^{th}-order presentations are treated here for completeness, though they are devoid of any real insight. Their associated equational logic is the 0^{th}-order simply-typed λ-calculus (Figure A.2), whose classifying category $\Lambda_0(S)$ is defined in Definition 3.2.

Proposition A.42. $\Lambda_0(S) \cong \mathbb{L}_0(S)$. □

$$\frac{}{\cdot \ \mathsf{ctx}}\ \textsc{empty} \qquad \frac{X \ \mathsf{type}}{x : X \ \mathsf{ctx}}\ \textsc{singleton} \qquad \frac{}{x : X \vdash x : X}\ \textsc{var}$$

$$\frac{}{B \ \mathsf{type}}\ (B \in S)\ \textsc{base}$$

Fig. A.2. The 0^{th}-order simply-typed λ-calculus on S.

Definition A.43. An S-sorted 0^{th}-order *signature* consists of a set O of operators and an arity function $|-| : O \to S$. A signature gives rise to a *syntactic category* Λ_O defined as the classifying category in Definition 3.2 with the following additional axiom schema.

$$\frac{}{\vdash o : \mathsf{B}} \ (o \in O, |o| = B) \ \textsc{op}$$

Definition A.44. An S-sorted 0^{th}-order *presentation* consists of a signature $(O, |-|)$, a set $E \subseteq \sum_{B \in S} \Lambda_O(1, B) \times \Lambda_O(1, B)$. Every presentation $\Sigma = (O, |-|, E)$ similarly gives rise to a syntactic category Λ_Σ defined as the syntactic category for the underlying signature with the following additional axiom schema. We denote by $Q_\Sigma : \Lambda_O \twoheadrightarrow \Lambda_\Sigma$ the quotient of Λ_O by the equations of Σ.

$$\frac{}{\vdash l \equiv r : \mathsf{B}} \ ((B, l, r) \in E) \ \textsc{eq}$$

Transliterations and translations coincide for 0^{th}-order algebraic theories, as it is not possible to form nontrivial compound terms.

Definition A.45. Let $\Sigma = (O, |-|, E)$ and $\Sigma' = (O', |-|', E')$ be 0^{th}-order presentations. An 0^{th}-order *transliteration/translation* from Σ to Σ' consists of a function $f : O \to O'$ such that $|f(o)|' = |o|$ for all $o \in O$, and such that, for all $B \in S$ and $(l, r) \in \Lambda_O(1, \mathsf{B})$, we have $Q_{\Sigma'}(f(l)) = Q_{\Sigma'}(f(r))$ if $Q_\Sigma(l) = Q_\Sigma(r)$.

S-sorted 0^{th}-order presentations and transliterations/translations form a category $\mathbf{Pre}_0(S) = \mathbf{Pres}_0(S)$, with composition and identities inherited from $\mathbf{Set}$.

Proposition A.46. $\mathbf{Pres}_0(S) \simeq \mathbf{Law}_0(S)$.

Proof. $\mathbf{Pres}_0(S)$ is elementarily equivalent to the category of S-indexed setoids, which in turn is equivalent to the category of S-indexed sets, which by Lemma 9.3 is equivalent to $\mathbf{Law}_0(S)$. □

Lemma 9.3. $\mathbf{Law}_0(S) \cong \mathbf{Set}^S$.

Proof. Follows trivially by considering the co-Yoneda embedding of the terminal object. □

Definition A.47. A *model* for an S-sorted 0^{th}-order algebraic theory $L : \mathbb{L}_0(S) \to \mathscr{L}$ in a category $\mathscr{C}$ with a terminal object is a terminal-object-preserving functor $M : \mathscr{L} \to \mathscr{C}$. A *map* of models from M to M' is a natural transformation $M \Rightarrow M'$. Models for $\mathscr{L}$ and their maps form a category $\mathbf{Mod}(L, \mathscr{C})$, functorial contravariantly in the first argument and covariantly in the second.

Remark A.1. *The notion of term algebra is trivial for 0^{th}-order algebraic theories, because every context in a 0^{th}-order algebraic theory is empty.*

Proposition A.48. $\mathbf{Law}_0(S)$ *is a coreflective subcategory of* $\mathbf{Law}_1(S)$.

Proof. The functor $\lceil - \rceil : \mathbf{Law}_0(S) \to \mathbf{Law}_1(S)$ sending a 0^{th}-order algebraic theory to its conservative cartesian completion is fully faithful, by the universal property of free cartesian completion. The functor $\lfloor - \rfloor : \mathbf{Law}_1(S) \to \mathbf{Law}_0(S)$ sending a first-order algebraic theory to its full subcategory on $S + 1$ is right adjoint to $\lceil - \rceil$, which follows directly from maps of theories being identity-on-objects. □

Corollary A.49. *Let $L : \mathbb{L}_n(S) \to \mathscr{L}$ be an n^{th}-order algebraic theory. The forgetful functor $L/\mathbf{Law}_n(S) \to \mathbf{Set}^S$ has a left adjoint.*

Proof. Direct by Theorem 5.9. □