Relative Monadicity

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Part I

Categories of structures

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$$\mathcal{D} \xrightarrow{U} \mathcal{E}$$

What does it mean for $\mathcal{D}$ to be a category of structured objects in $\mathcal{E}$?

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At minimum, we expect U to be faithful, but typically we are interested in something stronger.

Algebraic structure

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Categories of algebras for monads are very well understood, and so it is frequently useful to exhibit categories as monadic over well-behaved categories.

Algebras via extensions

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given an interpretation of X in E

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if we close X under the operations of T

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we can interpret all these new terms in E

satisfying two laws.

Arity

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However, in many situations we are only concerned with interpreting terms with small (e.g. finite) arity, for which the algebras for a monad are too restrictive.

Relative monads

A **relative monad** is a generalisation of a monad that captures the intuition that the operations may have restricted arity.

For $\mathcal{A} \xrightarrow{J} \mathcal{E}$, a J-relative monad comprises:

- a functor $\mathcal{A} \xrightarrow{T} \mathcal{E}$
- a natural transformation $J \overset{\eta}{\Rightarrow} T$
- for each $JX \xrightarrow{f} TY$, a $TX \xrightarrow{f^\dagger} TY$

subject to laws

Algebras for a relative monad

Given a $\mathcal{A} \xrightarrow{J} \mathcal{E}$-relative monad T, a T-algebra is an object $E \in \mathcal{E}$ equipped with, for each

$$\begin{tikzcd} TX \arrow[dr, dashed, "f^{\mathcal{E}}"] & \\ JX \arrow[u, "\eta_X"] \arrow[r, "f"'] & E \end{tikzcd}$$

satisfying two laws.

Algebraic theories

A suggestive class of examples arise in universal algebra. Monads relative to $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ are in bijection with algebraic theories.

Their algebras therefore only require extensions along functions $X \longrightarrow E$ with X finite, corresponding to finitarity of operations in universal algebra.

Categories of structures II

Given a category $\mathcal{E}$ and a functor $\mathcal{A} \xrightarrow{J} \mathcal{E}$, the categories of algebras for J-relative monads capture the intuition of categories whose objects are those of $\mathcal{E}$ equipped with operations with arity in $\mathcal{A}$, subject to equations.

Categories of structures II

Given a category $\mathcal{E}$ and a functor $\mathcal{A} \xrightarrow{J} \mathcal{E}$, the categories of algebras for J-relative monads capture the intuition of categories whose objects are those of $\mathcal{E}$ equipped with operations with arity in $\mathcal{A}$, subject to equations.

How can we characterise them?

Monadicity theorems

A functor $\mathcal{D} \xrightarrow{U} \mathcal{E}$ is monadic iff it admits a left adjoint and either:

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A functor $\mathcal{D} \xrightarrow{U} \mathcal{E}$ is monadic iff it admits a left adjoint and either:

Beck (c. 1966) U creates U-split coequalisers

or:

Paré (1971) U creates absolute colimits

Absolute colimits

Given a functor $\mathcal{X} \xrightarrow{D} \mathcal{E}$ admitting a colimit, $\operatorname{colim} D$ is <u>absolute</u> if either:

- it is preserved by <u>every</u> functor from $\mathcal{E}$
- it is preserved by the presheaf embedding $\mathcal{E} \hookrightarrow [\mathcal{E}^{op}, \mathbf{Set}]$

Relative monadicity

Given functors

$$\begin{tikzcd} & \mathcal{D} \arrow[dr, "U"] & \\ \mathcal{A} \arrow[rr, "J"'] & & \mathcal{E} \end{tikzcd}$$

we would like conditions under which U exhibits $\mathcal{D}$ as being the category of algebras for a J-relative monad.

Relative adjunctions

Given functors

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "R"] & \\ \mathcal{A} \arrow[ur, "L"] \arrow[rr, "J"'] & & \mathcal{E} \end{tikzcd}$$

we say L is left J-adjoint to R ($L \dashv_J R$) if

$$\mathcal{C}(Lx, y) \cong \mathcal{E}(Jx, Ry)$$

natural in x, y.

J-absolute colimits

Given a functor $\mathcal{A} \xrightarrow{J} \mathcal{E}$ and a functor $\mathcal{X} \xrightarrow{D} \mathcal{E}$ admitting a colimit, $\operatorname{colim} D$ is <u>J-absolute</u> if it is preserved by the restricted presheaf embedding

$$\mathcal{E} \hookrightarrow [\mathcal{E}^{op}, \mathrm{Set}] \xrightarrow{[J^{op}, \mathrm{Set}]} [\mathcal{A}^{op}, \mathrm{Set}]$$

The relative monadicity theorem

Consider functors:

$$\begin{tikzcd} & \mathcal{D} \arrow[d, "U"] \\ \mathcal{A} \arrow[r, "J"'] & \mathcal{E} \end{tikzcd}$$

where J is dense.

Theorem U is J-monadic iff it admits a left J-adjoint and creates J-absolute colimits.

Applications

Various frameworks for monad-theory correspondences have been proposed in recent years, the most popular being:

- Monads with arities
- Nervous monads

By the results of [Ark 22], these are all subsumed by relative monads.

Monadicity with arity & nervous monadicity

Consequently, the relative monadicity theorem exhibits monadicity theorems for monads with arities and nervous monads.

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Corollary Let $\mathcal{A} \xrightarrow{J} \mathcal{E}$ and $\mathcal{D} \xrightarrow{U} \mathcal{E}$ be functors. U exhibits $\mathcal{D}$ as the category of algebras for a monad with arities J / J-nervous monad iff U admits a left adjoint and creates J-absolute colimits.

Part II

When is the composite of two functors <u>monadic</u>?

Suppose we are given two functors

$$\mathcal{A} \xrightarrow{U_1} \mathcal{B} \xrightarrow{U_2} \mathcal{C}$$

and told they are both monadic.

We <u>cannot</u> conclude $\mathcal{A} \xrightarrow{U_2 U_1} \mathcal{C}$ is monadic (e.g. $\mathbf{Cat} \to \mathbf{Grph} \to \mathbf{Set}^2$).

When is the composite of two functors <u>monadic</u>?

There are various <u>heuristics</u> for detecting monadicity of composites (e.g. crude monadicity), but these only provide sufficient conditions.

We would prefer a precise characterisation.

The pasting law for pullbacks

Suppose we are given a diagram of morphisms as follows.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pb}"] & C \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

Then the outer rectangle is a pullback iff the left square is a pullback.

The pasting law for relative adjunctions

Suppose we are given a diagram of functors as follows.

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dr, "R"] & & \\
 & & \mathcal{D} \arrow[dr, "R'"] \arrow[d, phantom, "\dashv"] & \\
\mathcal{A} \arrow[uur, "L"] \arrow[urr, "L'"] \arrow[rrr, "J"'] & & & \mathcal{E}
\end{tikzcd}$$

The outer triangle is a relative adjunction iff the left triangle is a relative adjunction.

The pasting law for relatively monadic adjunctions

$$\begin{tikzcd}
 & & \mathcal{C} \arrow[dr, "R"] & & \\
 & & & \mathcal{D} \arrow[dr, "R'"] \arrow[d, phantom, "\dashv"] & \\
\mathcal{A} \arrow[uurr, "L"] \arrow[urrr, "L'"] \arrow[rrrr, "J"'] & & & {} & \mathcal{E}
\end{tikzcd}$$

Theorem If R' is J-monadic, then $R'R$ is J-monadic iff R is L'-monadic.

When is the composite of two functors monadic?

Suppose we are given two functors:

$$\mathcal{A} \xrightarrow{U_1} \mathcal{B} \xrightarrow{U_2} \mathcal{C}$$

Corollary If U_2 is monadic, then $U_2 U_1$ is monadic iff U_1 is F_2-monadic.

Useful corollaries

Corollary 1 Given adjunctions,

$$\begin{tikzcd} A \arrow[r, shift left, "U_1"] \arrow[r, phantom, "\top"] & B \arrow[l, shift left, "F_1"] \arrow[r, shift left, "U_2"] \arrow[r, phantom, "\top"] & \mathcal{C} \arrow[l, shift left, "F_2"] \end{tikzcd}$$

if U_2 and $U_2 U_1$ are monadic, so is U_1.

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Corollary 1 Given adjunctions,

$$\begin{tikzcd} A \arrow[r, shift left, "U_1"] \arrow[r, phantom, "\perp"] & B \arrow[l, shift left, "F_1"] \arrow[r, shift left, "U_2"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "F_2"] \end{tikzcd}$$

if U_2 and $U_2 U_1$ are monadic, so is U_1.

Corollary 2 For every morphism $S \Rightarrow T$ between finitary monads on a LFP category the induced functor $T\text{-Alg} \to S\text{-Alg}$ is monadic.

Summary

- Relative monads permit us to capture structures interpreting operations of restricted arity.
- We establish two monadicity theorems.
  - A Paré-style characterisation in terms of absolute colimits
  - A pasting law for relatively monadic adjunctions

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