

Relative Monadicity

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Part I

Categories of structures

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$$\mathcal{D} \xrightarrow{U} \mathcal{E}$$

What does it mean for \mathcal{D} to be a category of structured objects in \mathcal{E} ?

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At minimum, we expect U to be faithful, but typically we are interested in something stronger.

Algebraic structure

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Categories of algebras for monads are very well understood, and so it is frequently useful to exhibit categories as monadic over well-behaved categories.

Algebras via extensions

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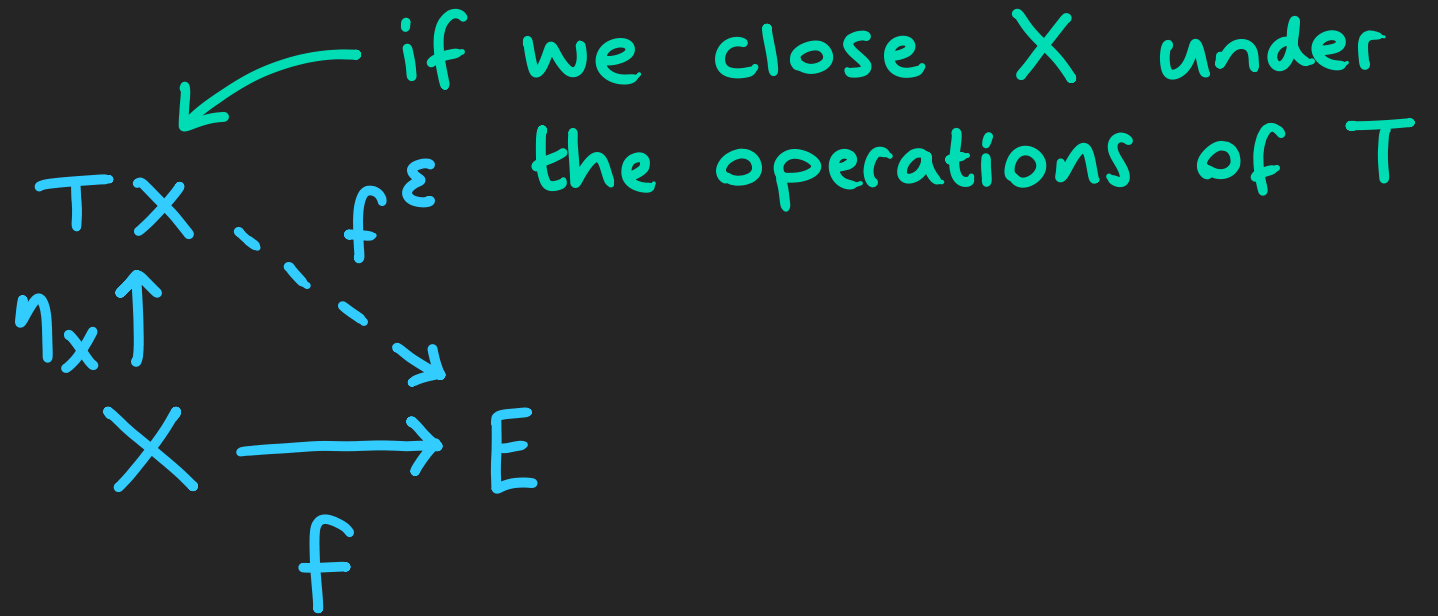
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we can interpret
all these new
terms in E

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Arity

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Because we must be able to extend morphisms $X \xrightarrow{f} E$ with arbitrary domain, E must potentially interpret operations with very large (e.g. infinitary) arity.

However, in many situations we are only concerned with interpreting terms with small (e.g. finite) arity, for which the algebras for a monad are too restrictive.

Relative monads

A relative monad is a generalisation of a monad that captures the intuition that the operations may have restricted arity.

For $\mathcal{A} \xrightarrow{\mathcal{J}} \mathcal{E}$, a \mathcal{J} -relative monad comprises:

- a functor $\mathcal{A} \xrightarrow{\mathcal{T}} \mathcal{E}$
 - a natural transformation $\mathcal{J} \xRightarrow{\eta} \mathcal{T}$
 - for each $\mathcal{J}X \xrightarrow{f} \mathcal{T}Y$, a $\mathcal{T}X \xrightarrow{f^{\dagger}} \mathcal{T}Y$
- } subject to laws

Algebras for a relative monad

Given a $\mathcal{A} \xrightarrow{T} \mathcal{E}$ -relative monad T , a T -algebra is an object $E \in \mathcal{E}$ equipped with, for each

$$\begin{array}{ccc} TX & \xrightarrow{f^E} & E \\ \eta_X \uparrow & \searrow & \\ JX & \xrightarrow{f} & E \end{array}$$

satisfying two laws.

Algebraic theories

A suggestive class of examples arise in universal algebra. Monads relative to $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ are in bijection with algebraic theories.

Their algebras therefore only require extensions along functions $X \rightarrow E$ with X finite, corresponding to finitariness of operations in universal algebra.

Categories of structures II

Given a category \mathcal{E} and a functor $\mathcal{A} \xrightarrow{T} \mathcal{E}$,
the categories of algebras for T -relative
monads capture the intuition of categories
whose objects are those of \mathcal{E} equipped with
operations with arity in \mathcal{A} , subject to
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How can we characterise them?

Monadicity theorems

A functor $D \xrightarrow{U} \mathcal{E}$ is monadic iff it admits a left adjoint and either:

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or:

Paré (1971) U creates absolute colimits

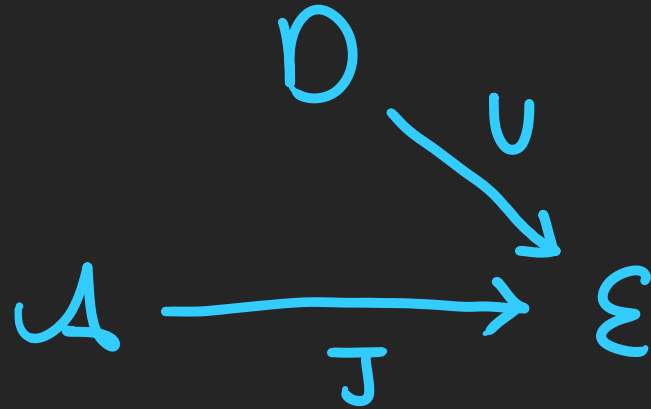
Absolute colimits

Given a functor $\mathcal{X} \xrightarrow{D} \mathcal{E}$ admitting a colimit, $\text{colim } D$ is absolute if either:

- it is preserved by every functor from \mathcal{E}
- it is preserved by the presheaf embedding $\mathcal{E} \hookrightarrow [\mathcal{E}^{\text{op}}, \text{Set}]$

Relative monadicity

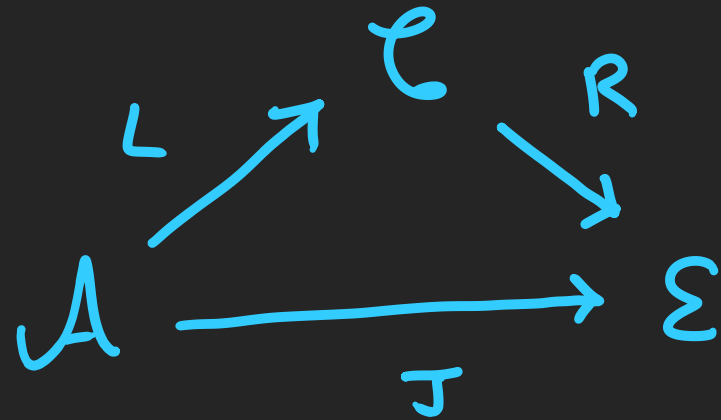
Given functors



we would like conditions under which U exhibits D as being the category of algebras for a J -relative monad.

Relative adjunctions

Given functors



we say L is left J -adjoint to R ($L \dashv_J R$) if

$$\mathcal{C}(Lx, y) \cong \mathcal{E}(Jx, Ry)$$

natural in x, y .

J-absolute colimits

Given a functor $\mathcal{A} \xrightarrow{J} \mathcal{E}$ and a functor $\mathcal{X} \xrightarrow{D} \mathcal{E}$ admitting a colimit, $\text{colim } D$ is J-absolute if it is preserved by the restricted presheaf embedding

$$\mathcal{E} \hookrightarrow [\mathcal{E}^{\text{op}}, \text{Set}] \xrightarrow{[J^{\text{op}}, \text{Set}]} [\mathcal{A}^{\text{op}}, \text{Set}]$$

The relative monadicity theorem

Consider functors:

$$\begin{array}{ccc} & \mathcal{D} & \\ & \searrow U & \\ \mathcal{A} & \xrightarrow{J} & \mathcal{E} \end{array}$$

where J is dense.

Theorem U is J -monadic iff it admits a left J -adjoint and creates J -absolute colimits.

Applications

Various frameworks for monad-theory correspondences have been proposed in recent years, the most popular being:

- Monads with arities
- Nervous monads

By the results of [Ark22], these are all subsumed by relative monads.

Monadicity with arity & nervous monadicity

Consequently, the relative monadicity theorem exhibits monadicity theorems for monads with arities and nervous monads.

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Corollary Let $A \xrightarrow{J} \mathcal{E}$ and $D \xrightarrow{U} \mathcal{E}$ be functors. U exhibits D as the category of algebras for a monad with arities J / J -nervous monad iff U admits a left adjoint and creates J -absolute colimits.

Part II

When is the composite of two functors
monadic?

Suppose we are given two functors

$$A \xrightarrow{U_1} B \xrightarrow{U_2} \mathcal{C}$$

and told they are both monadic.

We cannot conclude $A \xrightarrow{U_2 U_1} \mathcal{C}$ is
monadic (e.g. $\text{Cat} \rightarrow \text{Grph} \rightarrow \text{Set}^2$).

When is the composite of two functors
monadic?

There are various heuristics for detecting monadicity of composites (e.g. crude monadicity), but these only provide sufficient conditions.

We would prefer a precise characterisation.

The pasting law for pullbacks

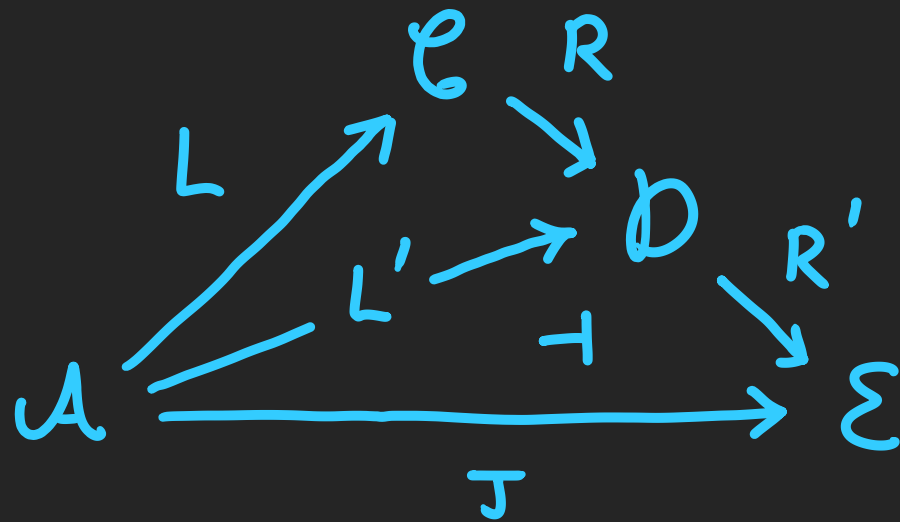
Suppose we are given a diagram of morphisms as follows.

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & \text{pb} & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

Then the outer rectangle is a pullback iff the left square is a pullback.

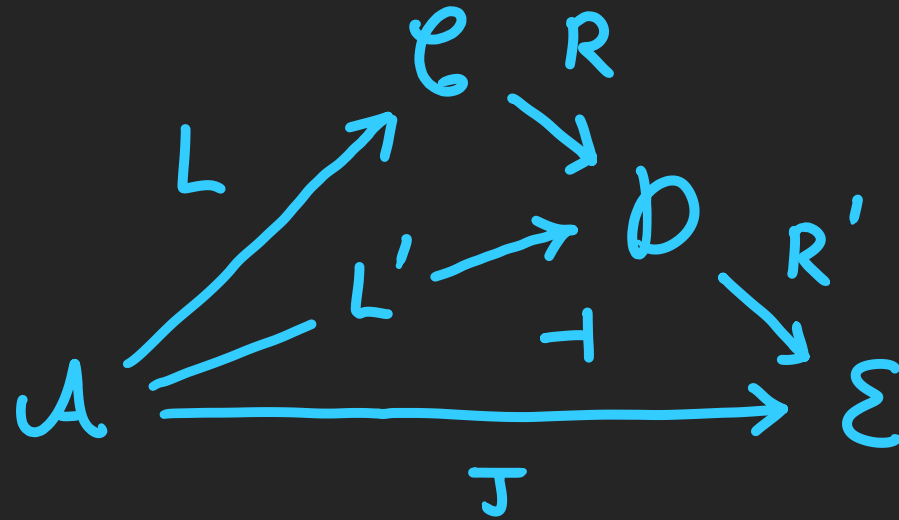
The pasting law for relative adjunctions

Suppose we are given a diagram of functors as follows.



The outer triangle is a relative adjunction
iff the left triangle is a relative adjunction.

The pasting law for relatively monadic adjunctions



Theorem If R' is J -monadic, then $R'R$ is J -monadic iff R is L' -monadic.

When is the composite of two functors
monadic?

Suppose we are given two functors:

$$A \xrightarrow{U_1} B \xrightarrow{U_2} C$$

Corollary If U_2 is monadic, then $U_2 U_1$
is monadic iff U_1 is F_2 -monadic.

Useful corollaries

Corollary 1 Given adjunctions,

$$A \begin{array}{c} \xrightarrow{U_1} \\ \xleftarrow[T_1]{F_1} \end{array} B \begin{array}{c} \xrightarrow{U_2} \\ \xleftarrow[T_2]{F_2} \end{array} \mathcal{C}$$

if U_2 and $U_2 U_1$ are monadic, so is U_1 .

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Corollary 2 For every morphism $S \Rightarrow T$

between finitary monads on a LFP category

the induced functor $T\text{-Alg} \rightarrow S\text{-Alg}$ is monadic.

Summary

- Relative monads permit us to capture structures interpreting operations of restricted arity.
- We establish two monadicity theorems.
 - A Paré-style characterisation in terms of absolute colimits
 - A pasting law for relatively monadic adjunctions

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