The nature of adjoint functor theorems

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The idea

The ubiquity of adjointness

> The slogan is "Adjoint functors arise everywhere."
>
> — Mac Lane [Mac98]

Adjoint functors are fundamental in category theory.

It is frequently useful to know that a functor admits an adjoint and, accordingly, one of the most well known theorems in category theory is the adjoint functor theorem [Fre64], which, together with its variants, which gives necessary and sufficient conditions for a functor to be left-adjoint.

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Theorem (Freyd)

A small-cocontinuous functor is left-adjoint if and only if it satisfies the solution set condition.

Definition

A functor $f \colon A \to B$ satisfies the solution set condition if, for every $b \in B$, the presheaf $B(f-, b) \colon A^{\mathrm{op}} \to \mathbf{Set}$ is *weakly multirepresentable*, i.e. there exists a small family of objects $(a_i)_{i \in I}$ in A and an epimorphism $\coprod_{i \in I} A(-, a_i) \twoheadrightarrow B(f-, b)$ in $[A^{\mathrm{op}}, \mathbf{Set}]$.

Why is the adjoint functor theorem true?

A suggestive reformulation

Theorem (Ulmer)

A small-cocontinuous functor is left-adjoint if and only if it is small-admissible.

Definition

A functor $f\colon A \to B$ is small-admissible if, for every $b \in B$, the presheaf $B(f-, b)\colon A^{\mathrm{op}} \to \mathbf{Set}$ is *small*, i.e. is a small colimit of representable presheaves.

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Ulmer's adjoint functor theorem consequently replaces the solution set condition in Freyd's adjoint functor theorem by the small-admissibility condition. Both conditions act to constrain the "size" of the functor in a certain sense.

The solution set condition versus small-admissibility

For the purposes of adjointness, the two conditions are equivalent.

Lemma (Ulmer)

A small-cocontinuous functor satisfies the solution set condition if and only if it is small-admissible.

We shall focus on Ulmer's formulation, for reasons that will become clear.

The three aspects of adjointness

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- Cocontinuity.
- Size.

What is the relationship between adjointness, cocontinuity, and size?

Relative adjointness

Definition (Ulmer)

Let $j\colon A \to E$, $\ell\colon A \to C$ and $r\colon C \to E$ be functors.

$$\begin{tikzcd} & C \arrow[dr, "r"] \arrow[d, phantom, "\dashv"] & \\ A \arrow[ur, "\ell"] \arrow[rr, "j"'] & {} & E \end{tikzcd}$$

Say that ℓ is left adjoint to $r\colon C \to E$ relative to j if there is an isomorphism of hom-sets

$$C(\ell a, c) \cong E(ja, rc)$$

natural in $a \in A$ and $c \in C$.

Identity-relative adjointness

A functor $f \colon A \to B$ is left-adjoint if and only if it is left-adjoint relative to the identity on A.

$$\begin{tikzcd}
 & B \arrow[dr, dashed] & \\
A \arrow[ur, "f"] \arrow[rr, equal] \arrow[rr, phantom, shift left=3ex, "\dashv"] & & A
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$$

The concept of relative adjointness thus subsumes the concept of adjointness.

Yoneda-relative adjointness

Every functor $f\colon A \to B$ between locally small categories is left-adjoint relative to the Yoneda embedding of A.

$$\begin{tikzcd} & B \arrow[dr, "B(f-_2{,}-_1)"] & \\ A \arrow[ur, "f"] \arrow[rr, "A(-_2{,}-_1)"'] \arrow[rr, phantom, shift left=4ex, "\dashv"] & & {[A^{\mathrm{op}}, \mathbf{Set}]} \end{tikzcd}$$

The right adjoint is given by the nerve functor (a.k.a. restricted Yoneda embedding), defined by:

$$b \mapsto (a \mapsto B(fa, b))$$

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The right adjoint is given by the nerve functor (a.k.a. restricted Yoneda embedding), defined by:

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Adjointness relative to the Yoneda embedding is therefore a tautology.

A spectrum of adjointnesses

More generally, we can consider adjointness of a functor $f\colon A \to B$ relative to a full subcategory $\tilde{A} \hookrightarrow [A^{\mathrm{op}}, \mathbf{Set}]$ of the category of presheaves. This provides a spectrum of notions of adjointness, ranging from ordinary adjointness to tautology.

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$$\begin{tikzcd} & B \arrow[dr, "B(f-_2,-_1)"] \arrow[d, phantom, "\dashv"] & \\ A \arrow[ur, "f"] \arrow[rr, "A(-_2,-_1)"'] & {} & \tilde{A} \end{tikzcd}$$

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- Taking $\tilde{A} := A$, we recover adjointness.
- Taking $\tilde{A} := [A^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{small}}$, the category of small presheaves on A, we recover small-admissibility.
- Taking $\tilde{A} := [A^{\mathrm{op}}, \mathbf{Set}]$, we have a tautology.

A reformulation of the adjoint functor theorem

The size condition of the adjoint functor theorem may therefore be expressed as a relative adjointness condition. Consequently, we may see the adjoint functor theorem as breaking adjointness down into cocontinuity and relative adjointness.

Theorem

A functor $f \colon A \to B$ is left-adjoint if and only if it is small-cocontinuous and left-adjoint relative to the restricted Yoneda embedding $A \to [A^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{small}}$.

A mild generalisation

The category of small presheaves $[A^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{small}}$ has a universal property: it is the free cocompletion of A under small colimits.

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Theorem (Tholen)

Let Φ be a class of colimits. A functor $f\colon A \to B$ is left-adjoint if and only if it is Φ-cocontinuous and Φ-admissible, i.e. left-adjoint relative to the free Φ-cocompletion $A \to [A^{\mathrm{op}}, \mathbf{Set}]_{\Phi}$.

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Why is this an improvement over Freyd's adjoint functor theorem? It explicates the strong connection between cocontinuity and admissibility: we can relax cocontinuity, so long as we strengthen admissibility.

Taking Φ to be the small colimits is in some sense the "strongest" assertion, as when $\Phi = \varnothing$, we get a tautology.

Examples of Φ-admissibility

Definition (Diers, $\Phi = \{$discrete colimits$\}$)

A functor $\ell \colon A \to C$ is left-multiadjoint when there exists a set I and an I-indexed family of functors $\{r_i \colon C \to A\}_{i \in I}$ together with an isomorphism of hom-sets, natural in $a \in A$ and $c \in C$.

$$C(\ell a, c) \cong \coprod_{i \in I} A(a, r_i c)$$

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Definition (Solian and Viswanathan, $\Phi = \{\text{finite colimits}\}$)

A functor $\ell \colon A \to C$ is left-pluriadjoint when there exists a finite category D and an D-indexed family of functors $\{r_i \colon C \to A\}_{i \in I}$ together with an isomorphism of hom-sets, natural in $a \in A$ and $c \in C$.

$$C(\ell a, c) \cong \operatorname{colim}_{d \in D} A(a, r_d c)$$

A family of AFTs

Φ	A functor is left-adjoint iff it ________.
$\varnothing$	is left-adjoint
Discrete colimits	preserves coproducts and is left-multiadjoint
Finite colimits	preserves finite colimits and is left-pluriadjoint
Small colimits	preserves small colimits and is small-admissible

An observation

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Note that left-adjointness is itself an admissibility property. Can we generalise $\varnothing$-admissibility to other notions of admissibility?

Theorem

Let Ψ and Φ be classes of colimits, for which Ψ-limits commute with Φ-colimits in $\mathbf{Set}$. A functor is Φ-admissible if and only if it is Ψ-cocontinuous and $(\Phi \circ \Psi)$-admissible.

Some examples

Some consequences of this general adjoint functor theorem follow.

- A functor is left-adjoint if and only if it preserves small colimits and is small-admissible [Ulm71].

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- A functor is left-adjoint if and only if it preserves small colimits and is small-admissible [Ulm71].
- A functor is left-multiadjoint if and only if it preserves coproducts and is small-admissible [Die77].
- A functor is left-pluriadjoint if and only if it preserves finite colimits and is small-admissible [SV90].

A formal understanding of the AFT

Versions of the adjoint functor theorem exist not just for ordinary functors, but also enriched functors, internal functors, indexed functors, and so on. Similarly, we should hope that this general admissibility theorem also holds in these settings.

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This is the motivation for formal category theory, which allows us to prove a theorem once, and to recover each of these settings as examples.

To prove such a theorem, we shall move to the context of 2-categories.

Lax-idempotent pseudomonads

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A lax-idempotent pseudomonad is a 2-dimensional notion of monad that captures the universal property of free cocompletions.

Definition ([Koc95; Zöb76])

A pseudomonad T is lax-idempotent if and only if every 1-cell between the underlying objects of two T-algebras has a unique lax T-algebra morphism structure.

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Definition

A lax-idempotent pseudomonad is *locally fully faithful* if, for each object a, the 1-cell $\eta_a \colon a \to Ta$ is representably fully faithful, i.e. if $\mathcal{K}[\eta_a, -] \colon \mathcal{K}[Ta, -] \to \mathcal{K}[a, -]$ is fully faithful.

Cocompleteness and cocontinuity

Definition

An object a is T-cocomplete if it is a T-algebra, equivalently if $\eta_a \colon a \to Ta$ admits a left adjoint with invertible counit.

Definition

A 1-cell $f \colon a \to b$ is T-cocontinuous if it is a pseudomorphism for T, equivalently if the unique 2-cell forming the lax morphism structure admits an inverse.

Φ-cocompletions

Let $\mathbb{V}$ be a nice monoidal category. For every class of $\mathbb{V}$-enriched colimits Φ, there is a locally fully faithful lax-idempotent pseudomonad on the 2-category $\mathbb{V}$-$\mathbf{CAT}$ of (large) $\mathbb{V}$-categories sending each $\mathbb{V}$-category to its cocompletion under Φ-colimits, given by $A \mapsto [A^{\mathrm{op}}, \mathbb{V}]_\Phi$.

A pseudoalgebra for this pseudomonad is a Φ-cocomplete $\mathbb{V}$-category, and a functor is a pseudomorphism if and only if it is Φ-cocontinuous.

Admissibility

Definition ([BF99])

A 1-cell $f\colon a \to b$ is T-admissible if Tf admits a right adjoint.

$$\begin{tikzcd} Ta \arrow[r, shift left, "Tf"] \arrow[r, phantom, "\perp"] & Tb \arrow[l, shift left, dashed, "f^T"] \end{tikzcd}$$

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Example

When T corresponds to the cocompletion of a $\mathbb{V}$-category under some class Φ of colimits, then T-admissibility corresponds to relative adjointness, i.e. Φ-admissibility in the ordinary sense.

$$\begin{tikzcd} & B \arrow[dr, dashed] \arrow[d, phantom, "\dashv"] & \\ A \arrow[ur, "f"] \arrow[rr] & {} & {[A^{\mathrm{op}}, \mathbb{V}]_\Phi} \end{tikzcd}$$

Pseudodistributive laws

A pseudodistributive law between two pseudomonads is a 2-dimensional notion of distributive law.

Definition

Let R and I be lax-idempotent pseudomonads on 2-category $\mathcal{K}$. R *distributes over* I if there exists a lifting $\tilde{I}$ of I to the 2-category R-$\mathbf{Alg}$ of R-algebras.

$$\begin{tikzcd} R\text{-}\mathbf{Alg} \arrow[r, "\tilde{I}"] \arrow[d, "U_R"'] & R\text{-}\mathbf{Alg} \arrow[d, "U_R"] \\ \mathcal{K} \arrow[r, "I"'] & \mathcal{K} \end{tikzcd}$$

Distributivity via commutativity

Let Ψ and Φ be classes of $\mathbb{V}$-enriched colimits, for which Ψ-limits commute with Φ-colimits in $\mathbb{V}$. Then the pseudomonad induced by Ψ distributes over the pseudomonad induced by Φ.

Ψ	Φ	Φ-admissibility
$\varnothing$	Small weights	Small-admissibility
Connected weights	Discrete weights	Multiadjointness
Finite weights	Filtered weights	Pluriadjointness
Small weights	$\varnothing$	Adjointness

Main result

Theorem

Let R and I be lax-idempotent pseudomonads for which R distributes over I. Suppose that I is locally fully faithful. A 1-cell $f\colon a \to b$ between R-cocomplete objects is I-admissible if and only if it is R-cocontinuous and IR-admissible.

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Corollary

Let R be a lax-idempotent pseudomonad. A 1-cell $f\colon a \to b$ between R-cocomplete objects is left-adjoint if and only if it is R-cocontinuous and R-admissible.

The AFT for locally presentable categories

Theorem

Let $f\colon A \to B$ be a functor between locally presentable categories.

1. *f is left-adjoint if and only if it is small-cocontinuous.*
2. *f is right-adjoint if and only if it is small-continuous and accessible.*

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Proof.

1. Every functor between locally presentable categories is small-admissible.

□

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Proof.

1. Every functor between locally presentable categories is small-admissible.
2. A functor between accessible categories is small-coadmissible if and only if it is accessible [LT23].

□

Summary

- The adjoint functor theorem fundamentally expresses the relationship between adjointness, cocontinuity, and admissibility.
- These properties are captured respectively by the notions of adjointness in a 2-category, and cocontinuity and admissibility for a lax-idempotent pseudomonad.
- Adjointness is itself an admissibility property, and the adjoint functor theorem can be generalised to this context.
- The relationship between admissibility and cocontinuity is mediated by a pseudodistributive law between lax-idempotent pseudomonads.

You can read our (12-page) preprint on arXiv:

Adjoint functor theorems for lax-idempotent pseudomonads [ADL23]

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