The pullback theorem for relative monads

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TSEM, March 2024

Motivation

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The purpose of this talk is to explain the precise relationship between algebras and free algebras. While this is a simple question, it leads us down a path lined with insights into the nature of categorical algebra.

Presentations of algebras

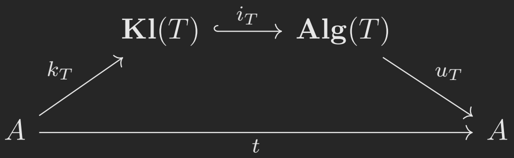
Every algebra is a quotient of free algebras. Explicitly, for a monad $T = (t, \mu, \eta)$ on a category A, and T-algebra (a, α), the following diagram exhibits a coequaliser in the category of T-algebras.

$$tta \underset{\mu_a}{\overset{t\alpha}{\rightrightarrows}} ta \xrightarrow{\alpha} a$$

Conceptually, this observation captures the intuition that we may present an algebra by describing its generating operators, together with equations that identify some of the resulting generated terms.

The comparison functor

Given a monad T, we may form the category $\mathbf{Alg}(T)$ of all T-algebras, and the category $\mathbf{Kl}(T)$ of free T-algebras. The category of free T-algebras embeds into the category of all T-algebras, exhibiting a fully faithful comparison functor $i_T \colon \mathbf{Kl}(T) \hookrightarrow \mathbf{Alg}(T)$.



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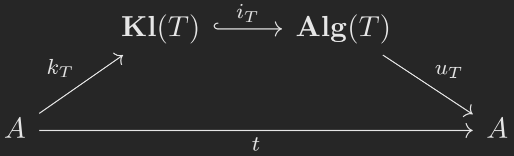

How does this observation relate to presentations of algebras?

The pullback theorem for monads

Let T be a monad on a category A. The following diagram forms a pullback of categories [Lin69].

$$\begin{array}{ccc}
\mathbf{Alg}(T) & \hookrightarrow & [\mathbf{Kl}(T)^{\mathrm{op}}, \mathbf{Set}] \\
{\scriptstyle u_T}\downarrow & \lrcorner & \downarrow{\scriptstyle [{k_T}^{\mathrm{op}}, \mathbf{Set}]} \\
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When A is small, $[\mathbf{Kl}(T)^{\mathrm{op}}, \mathbf{Set}]$ is the free cocompletion of $\mathbf{Kl}(T)$, and so the pullback theorem states that the category of T-algebras is a certain cocompletion of the category of free T-algebras. This generalises the earlier observation about quotients of free algebras from individual algebras to the entire category of algebras.

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Furthermore, the unlabelled functor is isomorphic to the nerve $n_{i_T} := \mathbf{Alg}(T)(i_T{-_2}, {-_1})$ of the comparison functor.

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1. how the pullback theorem for monads generalises to a pullback theorem for relative monads;
2. the conceptual explanation for the result;
3. some applications of the pullback theorem for relative monads.

Overview

1. Distributors and double categories
2. Monads and loose-monads
3. Relative monads
4. Categories of free algebras
5. The pullback theorem
6. Consequences

Distributors and double categories

On the nature of category theory

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- Categories.
- Functors.
- Natural transformations.

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- Categories.
- Functors.
- Natural transformations.

However, there are many fundamental concepts in category theory that cannot be defined with just these concepts (e.g. weighted colimits, pointwise extensions, density, full faithfulness, etc.). To capture these concepts, we need one more building block.

- Distributors.

Distributors

Definition 1 (Bénabou)

Let A and B be categories. A distributor (a.k.a. profunctor or (bi)module) $A \nrightarrow B$ is a functor $B^{\mathrm{op}} \times A \to \mathbf{Set}$.

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A distributor $p \colon A \nrightarrow B$ may be thought of as a categorified notion of relation, i.e. a function $B \times A \to \{\bot, \top\}$. A helpful intuition is to think of the elements of $p(b, a)$, for each $b \in |B|$ and $a \in |A|$, as heteromorphisms from b to a. Functoriality of p then ensures that heteromorphisms in p are closed under precomposition by morphisms in B and under postcomposition by morphisms in A.

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(In more general settings, such as enriched category theory, distributors may not be defined in terms of functors, which is why we view them as a fundamentally separate concept. This is particularly crucial for formal category theory.)

Yoneda embedding as a distributor

As suggested by the definition of a distributor, to every category A there is a canonical endo-distributor on A, given by the homomorphisms of A.

Example 2

Let A be a (locally small) category. The hom-sets of A form an identity distributor $A(1,1)\colon A \nrightarrow A$, defined by

$$A(1,1)(a,a') := A(a,a')$$

(Note that the hom-set functor $A^{\mathrm{op}} \times A \to \mathbf{Set}$ is the uncurried form of the Yoneda embedding $A \to \mathbf{Set}^{A^{\mathrm{op}}}$.)

Representable and corepresentables

Every functor $f\colon A \to B$ induces two distributors.

Example 3

A representable distributor $B(1,f)\colon A \nrightarrow B$, defined by

$$B(1,f)(b,a) := B(b,fa)$$

Example 4

A corepresentable distributor $B(f,1)\colon B \nrightarrow A$, defined by

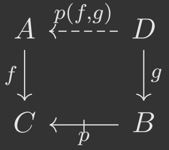
$$B(f,1)(a,b) := B(fa,b)$$

Restriction

The representable and corepresentable distributors associated to a functor are special cases of the following construction.

Example 5

Given every diagram of the following form,



there is a distributor $p(f,g)\colon A \nrightarrow D$, defined by

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Universal properties via distributors

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Proposition 6

Let $\ell \colon A \to B$ and $r \colon B \to A$ be functors. Then $\ell \dashv r$ if and only if there is an isomorphism of distributors:

$$B(\ell, 1) \cong A(1, r) \colon B \nrightarrow A$$

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Proposition 6

Let $\ell\colon A \to B$ *and* $r\colon B \to A$ *be functors. Then* $\ell \dashv r$ *if and only if there is an isomorphism of distributors:*

$$B(\ell, 1) \cong A(1, r)\colon B \nrightarrow A$$

As a consequence, we are able to define (weighted) limits and colimits, pointwise extensions, density, full faithfulness, etc. in terms of distributors.

Distributors versus presheaf categories

For ordinary categories, we can alternatively define these concepts in terms of presheaf categories. That is, for locally small categories A and B, a distributor $A \nrightarrow B$ is equivalently a functor $A \to [B^{\mathrm{op}}, \mathbf{Set}]$ by currying. However, this is not possible in general for other flavours of category theory, such as enriched category theory, where presheaf categories may not exist.

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Furthermore, using distributors allows us not to worry about size issues (e.g. taking presheaf categories of large categories).

We shall see more advantages to reasoning using distributors, in connection to the theory of monads, later in the talk.

The structure that distributors form

There are many examples of "category-like structures" that are of interest in category theory. Some examples are enriched categories, internal categories, fibred categories, indexed categories, monoidal categories, and so on.

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Axiomatising the structure of categories, functors, and natural transformations led early category theorists to the concept of 2-category [God58; Bén65; Mar65]. Enriched categories, internal categories, fibred categories, and so on, all form 2-categories.

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However, as we have mentioned, to carry out a significant amount of category theory, we also need to consider distributors. What structure, then, do categories, functors, distributors, and natural transformations form?

Virtual double categories

A virtual double category [Bur71] has a collection of objects, a collection of tight-cells $\bullet \to \bullet$ between objects, a collection of loose-cells $\bullet \nrightarrow \bullet$ between objects, and a collection of 2-cells of the following shape.

$$\begin{array}{ccccccccc}
A_0 & \xleftarrow{p_1} & A_1 & \xleftarrow{p_2} & \cdots & \xleftarrow{p_{n-1}} & A_{n-1} & \xleftarrow{p_n} & A_n \\
{\scriptstyle f}\downarrow & & & & \phi & & & & \downarrow{\scriptstyle g} \\
B_0 & & & & \xleftarrow[q]{\qquad\qquad\qquad} & & & & B_n
\end{array}$$

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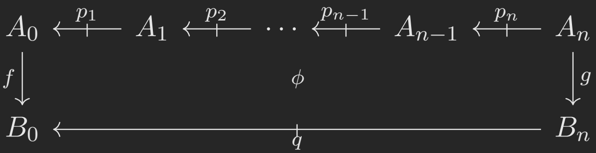

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While the definition of a virtual double category may at first appear intimidating, in practice it quickly becomes intuitive to reason about them, for instance by using a string diagram calculus.

Natural transformations

A natural transformation of the form

$$
\begin{array}{ccccccccc}
A_0 & \overset{p_1}{\longleftarrow\!\!\!+} & A_1 & \overset{p_2}{\longleftarrow\!\!\!+} & \cdots & \overset{p_{n-1}}{\longleftarrow\!\!\!+} & A_{n-1} & \overset{p_n}{\longleftarrow\!\!\!+} & A_n \\
f\downarrow & & & & \phi & & & & \downarrow g \\
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\end{array}
$$

comprises a family of functions

$$\phi_{x_0,\ldots,x_n} \colon p_1(x_0, x_1) \times \cdots \times p_n(x_{n-1}, x_n) \to q(fx_0, gx_n)$$

for $x_0 \in |A_0|, \ldots, x_n \in |A_n|$, satisfying certain naturality laws.

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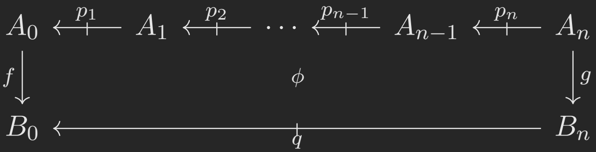

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In other words, a natural transformation essentially tells us how to compose a chain of heteromorphisms.

When $n = 0$ and q is trivial, this is exactly the usual notion of natural transformation $\phi \colon f \Rightarrow g$ between functors.

The virtual double category of categories

The motivating example of a virtual double category is $\mathbb{C}\mathbf{at}$, the virtual double category whose objects are locally small categories, whose tight-cells are functors, whose loose-cells are distributors, and whose 2-cells are natural transformations.

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Other examples of virtual double categories include the virtual double categories $\mathbb{V}$-$\mathbb{C}\mathbf{at}$, of categories enriched in a monoidal category $\mathbb{V}$; $\mathbb{C}\mathbf{at}(\mathbb{E})$, of categories internal to a finitely complete category $\mathbb{E}$; as well as virtual double categories of fibred categories, indexed categories, monoidal categories, and so on.

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In fact, these virtual double categories are particularly well-behaved, having identity loose-cells, and restrictions of loose-cells along tight-cells. Such virtual double categories are known as virtual equipments.

Monads and loose-monads

Monads in a virtual double category

Since a virtual double category has two kinds of morphisms (tight and loose), there are two kinds of monads we can consider inside a virtual double category.

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A (tight) monad comprises a tight-cell $t\colon A \to A$, and 2-cells $\mu\colon tt \to t$ and $\eta\colon 1_A \Rightarrow t$ satisfying associativity and unitality axioms.

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle tt}\downarrow & \mu & \downarrow{\scriptstyle t} \\ A & \overset{\shortmid}{=} & A \end{array} \qquad\qquad \begin{array}{ccc} A & = & A \\ \| & \eta & \downarrow{\scriptstyle t} \\ A & \overset{\shortmid}{=} & A \end{array}$$

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A loose-monad comprises a loose-cell $t\colon A \nrightarrow A$, and 2-cells $\mu\colon t, t \to t$ and $\eta\colon \ \Rightarrow t$ satisfying associativity and unitality axioms.

$$
\begin{array}{ccccc}
A & \xleftarrow{t} & A & \xleftarrow{t} & A \\
\| & & \mu & & \| \\
A & & \xleftarrow[t]{\qquad} & & A
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & = & A \\
\| & \eta & \| \\
A & \xleftarrow[t]{} & A
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Monads and loose-monads in $\mathbb{C}\mathbf{at}$

A monad in $\mathbb{C}\mathbf{at}$ is simply an ordinary monad, i.e. a functor $t \colon A \to A$ equipped with natural transformations $\mu \colon tt \Rightarrow t$ and $\eta \colon 1_A \Rightarrow t$ satisfying associativity and unitality axioms.

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A loose-monad (a.k.a. promonad) in $\mathbb{C}\mathbf{at}$ comprises

1. a distributor $p\colon A \nrightarrow A$;
2. for each $f\colon x \to y$ in A, an element $\eta_f \in p(x, y)$;
3. for $f \in p(x, y)$ and $g \in p(y, z)$, an element $(f \mathbin{;} g) \in p(x, z)$;

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Indeed, every category A induces a canonical loose-monad $A(1, 1)$, whose underlying distributor is given by the hom-sets of A, whose unit is trivial, and whose multiplication is given by composition in A.

The collapse of a loose-monad

In fact, every loose-monad p induces a category $«p»$, the collapse of p, defined by

$$|«p»| := |A| \qquad\qquad «p»(x, y) := p(x, y)$$

The collapse is equipped with an identity-on-objects functor $\underline{\upsilon}_p \colon A \to «p»$, which sends $f \colon x \to y$ in A to $\eta_f \colon x \to y$ in $«p»$.

Relative monads

Monoids in multicategories

A multicategory [Lam69] is a generalisation of a category in which we permit morphisms with multiary domain (analogous to the 2-cells in a virtual double category).

We can define monoids internal to any multicategory, generalising the notion of monoid internal to a monoidal category.

Definition 7

Let $\mathbf{M}$ be a multicategory. A monoid in $\mathbf{M}$ comprises

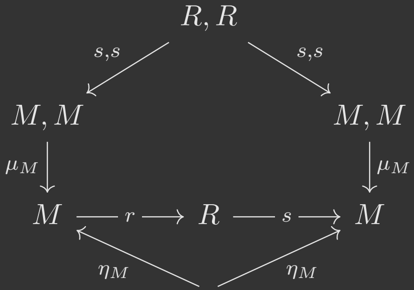
1. an object M;
2. a multimorphism $\mu \colon M, M \to M$;
3. a multimorphism $\eta \colon \to M$,

satisfying associativity and unitality axioms.

Monoid sections I

Definition 8

Let $\mathbf{M}$ be a multicategory and let (M, μ_M, η_M) be a monoid in $\mathbf{M}$. An (M, μ_M, η_M)-section comprises a section–retraction pair $s\colon R \rightleftarrows M\ :r$ rendering the following diagram commutative.



Monoid sections II

Conceptually, a monoid section is a retract R of the carrier of a monoid M, for which the section morphism $s\colon R \to M$ satisfies the laws to be a monoid morphism, with respect to "tentative monoid structure" on R.

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It turns out that this suffices for R to itself be a monoid, whose multiplication and unit are inherited from M.

Proposition 9

Let $\mathbf{M}$ be a multicategory and let (M, μ_M, η_M) be a monoid in $\mathbf{M}$. An (M, μ_M, η_M)-section (R, s, r) endows R with a unique monoid structure such that s is a monoid morphism.

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Why is this interesting? It turns out that we can characterise relative monads in this way.

Relative monads as monoid sections

Definition 10

A relative monad comprises a functor $t\colon A \to E$ along with a t-corepresentable $E(t,t)$-section.

Unwrapping this definition, we obtain the classical definition of a relative monad [ACU10], i.e. that a relative monad comprises

1. a functor $j\colon A \to E$, the *root*;
2. a functor $t\colon A \to E$, the *carrier*;
3. a natural transformation $\eta\colon j \Rightarrow t$, the *unit*;
4. a natural transformation $\dagger\colon E(j,t) \Rightarrow E(t,t)$, the *extension operator*,

satisfying unitality and associativity axioms.

When $j = 1$, this is equivalent to the usual definition of a monad.

Examples of relative monads

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- Cocontinuous monads on cocompletions (e.g. finitary monads on locally finitely presentable categories).

Examples of relative monads

Relative monads are abundant in category theory.

- Monads.
- Partial monads.
- Graded monads [MU22].
- Cocontinuous monads on cocompletions (e.g. finitary monads on locally finitely presentable categories).
- Monads arising from monad–theory correspondences [Ark22].

The loose-monad associated to a relative monad

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One reason is that we immediately obtain the following observation.

Corollary 11

Let T be a j-relative monad. The distributor $E(j,t) \colon A \nrightarrow A$ is equipped with the structure of a loose-monad $E(j,T)$, and $\dagger \colon E(j,t) \Rightarrow E(t,t)$ is a loose-monad morphism.

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Why is this nice? As we will see in the remainder of the talk, a relative monad T and its associated loose-monad $E(j,T)$ are strongly connected. The presentation of relative monads in terms of monoid sections emphasises this connection: in some sense, we can view $E(j,T)$ as encapsulating the fundamental structure of T.

(Free) algebras via loose-monads

If we can capture the structure of relative monads via their associated loose-monads, it is natural to ask whether we might also capture the algebras and free algebras for a relative monad T in terms of its associated loose-monad $E(j, T)$.

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As we shall see, this question leads inevitably to the pullback theorem.

Categories of free algebras

Kleisli categories for relative monads

Just as for non-relative monads, there are two important categories associated to every relative monad.

Definition 12 ([ACU10])

Let $j \colon A \to E$ be a functor and let T be a j-relative monad. The Kleisli category of T is the category $\mathbf{Kl}(T)$ defined by

$$|\mathbf{Kl}(T)| := |A|$$
$$\mathbf{Kl}(T)(x, y) := E(jx, ty)$$

with identities and composition given as in the Kleisli category for a monad.

This is equipped with an inclusion functor $k_T \colon A \to \mathbf{Kl}(T)$.

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This definition may look reminiscent of an earlier one...

Kleisli categories via collapse

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Let T be a j-relative monad. The Kleisli category of T is precisely the collapse of the loose-monad $E(j, T)$.

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- The universal property of a collapse is *stronger* than that typically associated with a Kleisli category (namely an opalgebra object). This allows us to prove stronger theorems than we would otherwise be able to prove.

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- The universal property of a collapse is *stronger* than that typically associated with a Kleisli category (namely an opalgebra object). This allows us to prove stronger theorems than we would otherwise be able to prove.
- It justifies our perspective that $E(j, T)$ represents T in a suitable sense, since we can recover T from $\mathbf{Kl}(T)$ via its associated relative adjunction.

The pullback theorem

Categories of algebras

Definition 14 ([ACU10])

Let $j \colon A \to E$ be a functor and let T be a j-relative monad. A T-algebra is an object $e \in E$ equipped with a natural transformation $\rtimes \colon E(j, e) \Rightarrow E(t, e)$ that is compatible with the unit and extension operator of T.

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The category of algebras of T is the category $\mathbf{Alg}(T)$ whose objects are T-algebras and whose morphisms are morphisms in E preserving the algebra structure.

This is equipped with a forgetful functor $u_T\colon \mathbf{Alg}(T) \to E$.

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This is equipped with a forgetful functor $u_T\colon \mathbf{Alg}(T) \to E$.

When $j = 1$, this is equivalent to the usual definition of the category of algebras for a monad.

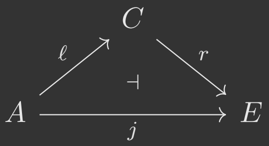
Relative adjunctions

The concept of relative adjunction is a generalisation of the concept of adjunction, where the domain of the left adjoint is permitted to be different to the codomain of the right adjoint.

Definition 15 ([Ulm68])

A relative adjunction comprises

1. a functor $j \colon A \to E$, the *root*;
2. a functor $\ell \colon A \to C$, the *left relative adjoint*;
3. a functor $r \colon C \to E$, the *right relative adjoint*;
4. an isomorphism of the form $C(\ell, 1) \cong E(j, r)$.



Examples of relative adjunctions

Relative adjunctions are abundant in category theory.

- Adjunctions.

Examples of relative adjunctions

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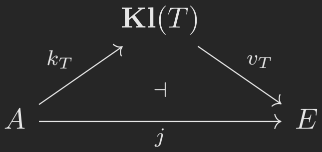
Examples of relative adjunctions

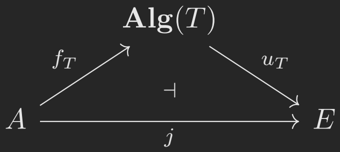
Relative adjunctions are abundant in category theory.

- Adjunctions.
- Partial adjunctions.
- Multi-adjunctions.
- Weighted colimits.
- Nerves.
- Algebraic theories and their various generalisations [Die74; Ark22].

Kleisli and Eilenberg–Moore relative adjunctions

Just as for non-relative monads, the Kleisli category and category of algebras associated to a relative monad T form relative adjunctions, which induce the relative monad T by composing the left relative adjoint with the right relative adjoint.





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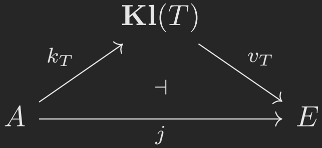

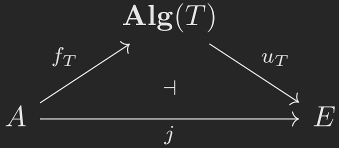

Furthermore, these relative adjunctions satisfy universal properties amongst resolutions of T – i.e. relative adjunctions inducing T – which induce a canonical comparison functor $i_T \colon \mathbf{Kl}(T) \to \mathbf{Alg}(T)$.

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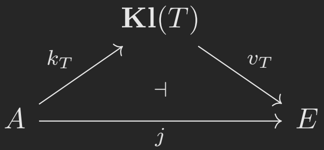

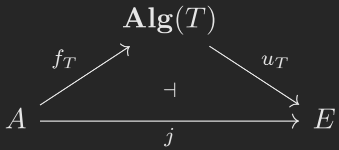

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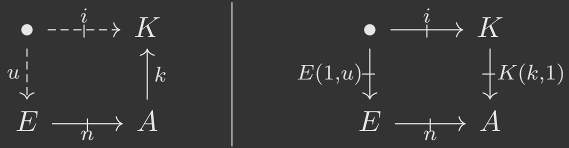
As we shall see, this comparison functor exhibits a stronger universal property than is implied simply by the universal properties of $\mathbf{Kl}(T)$ or $\mathbf{Alg}(T)$ individually.

Semanticisers

Definition 16

A semanticiser of a distributor $n \colon E \nrightarrow A$ and a functor $k \colon A \to K$ comprises a span of a distributor and functor, as on the left, such that the diagram on the right commutes,



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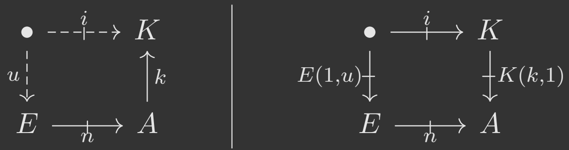


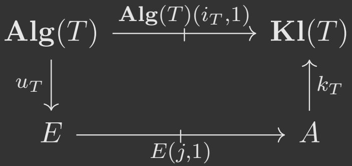
i.e. such that $i(k,1) = n(1,u)$, that is universal in the evident sense.

A semanticiser is a kind of equipment-theoretic limit.

The semanticiser theorem

Theorem 17

Let $j\colon A \to E$ be a dense functor and let T be a j-relative monad. Up to isomorphism, the following diagram is a semanticiser.



This is striking, because it identifies a nontrivial universal property, mediated by the comparison functor, that connects $\mathbf{Kl}(T)$ and $\mathbf{Alg}(T)$.

Presheaf categories

Definition 18

Let A be a small category. The category of presheaves on A is the functor category $\widehat{A} := [A^{\mathrm{op}}, \mathbf{Set}]$. Denote by よ$_A \colon A \to \widehat{A}$ the Yoneda embedding, defined by

$$よ_A(a) := A(-, a)$$

Presheaf categories

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$$よ_A(a) := A(-, a)$$

We can reformulate the Yoneda lemma in terms of a universal property involving distributors.

Lemma 19

The Yoneda embedding $\widehat{A}(よ_A, 1)$ induces a bijection between functors $B \to \widehat{A}$ and distributors $B \nrightarrow A$.

Nerves

Definition 20

For any functor $f\colon A \to B$ from a small category, there is a functor $n_f\colon B \to \widehat{A}$, the nerve of f, defined by

$$n_f(b) := B(f-, b)$$

The nerve is the functor corresponding, via the bijection on the previous slide, to the corepresentable distributor $B(f, 1)\colon B \nrightarrow A$.

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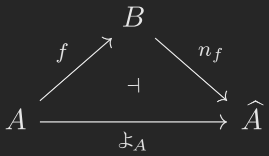
The nerve of f is right relative adjoint to f.

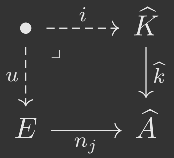


The pullback theorem I

In the presence of categories of presheaves, we may reformulate the universal property of a semanticiser into one involving only functors (rather than distributors). This allows us to easily give a concrete description.

Theorem 21

In $\mathbb{C}\mathbf{at}$*, the semanticiser of* $E(j, 1) \colon E \nrightarrow A$ *and* $A \xrightarrow{k} K$*, where* A *and* K *are small, is given by the following pullback.*



The pullback theorem II

Corollary 22

Let $j \colon A \to E$ be a dense functor and let T be a j-relative monad. The following diagram is a pullback in $\mathbb{C}\mathbf{at}$.

$$\begin{array}{ccc} \mathbf{Alg}(T) & \hookrightarrow & \widehat{\mathbf{Kl}(T)} \\ {\scriptstyle u_T}\downarrow & \lrcorner & \downarrow{\scriptstyle \widehat{k_T}} \\ E & \underset{n_j}{\hookrightarrow} & \widehat{A} \end{array}$$

Consequently, the comparison functor $i_T \colon \mathbf{Kl}(T) \hookrightarrow \mathbf{Alg}(T)$ is dense.

The algebras for a relative monad may thus be seen as a free cocompletion of the free algebras.

Non-relative case

When $j = 1$, we recover the pullback theorem for non-relative monads.

Theorem 23 (Linton)

Let T be a monad on a category A. The following diagram is a pullback in $\mathbb{C}\mathbf{at}$.

$$\begin{array}{ccc}
\mathbf{Alg}(T) & \hookrightarrow & \widehat{\mathbf{Kl}(T)} \\
{\scriptstyle u_T}\downarrow & & \downarrow{\scriptstyle \widehat{k_T}} \\
A & \underset{よ_A}{\hookrightarrow} & \widehat{A}
\end{array}$$

Consequences

Algebraic theories and relative monads

One of our motivations for studying relative monads is their connection to algebraic theories and their generalisations.

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Definition 24 (Lawvere)

Denote by $\mathbb{F}$ the free category with strict finite coproducts on a single object. A finitary algebraic theory is an identity-on-objects functor from $\mathbb{F}$ that preserves finite coproducts.

Algebraic theories and relative monads

One of our motivations for studying relative monads is their connection to algebraic theories and their generalisations.

Definition 24 (Lawvere)

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Theorem 25

There is an isomorphism between the category of finitary algebraic theories and the category of $(\mathbb{F} \to \mathbf{Set})$*-relative monads.*

More specifically, every algebraic theory is the Kleisli inclusion of a relative monad [Ark22].

Models and algebras

The pullback theorem establishes that the correspondence between algebraic theories and relative monads commutes with the process of taking models and algebras respectively.

Corollary 26

Let $\ell \colon \mathbb{F} \to L$ be a finitary algebraic theory. The category of algebras for the induced relative monad is given by the following pullback in $\mathbb{C}\mathbf{at}$.

$$\begin{array}{ccccc}
\mathbf{Alg}(T_\ell) & \simeq & \mathbf{Cart}(L^{\mathrm{op}}, \mathbf{Set}) & \hookrightarrow & \widehat{L} \\
{\scriptstyle u_{T_\ell}}\downarrow & & \downarrow{\scriptstyle \mathbf{Cart}(\ell^{\mathrm{op}}, \mathbf{Set})} \quad \lrcorner & & \downarrow{\scriptstyle \widehat{\ell}} \\
\mathbf{Set} & \simeq & \mathbf{Cart}(\mathbb{F}^{\mathrm{op}}, \mathbf{Set}) & \hookrightarrow & \widehat{\mathbb{F}}
\end{array}$$

Cocontinuous monads and relative monads

Another motivation for studying relative monads is their connection to cocontinuous monads.

Theorem 27

Let Φ be a class of colimits. There is an equivalence between the category of Φ-cocontinuous monads on $\Phi(A)$ and the category of $(A \to \Phi(A))$-relative monads, and this commutes with the process of taking algebras.

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Corollary 28

Let A be a small, finitely cocomplete category. There is an equivalence between the category of finitary monads on $\mathbf{Ind}(A)$ and the category of $(A \to \mathbf{Ind}(A))$-relative monads, and this commutes with the process of taking algebras.

Locally presentable categories of algebras

Corollary 29

Let T be a finitary monad on a locally finitely presentable category. Then its category of algebras is also locally finitely presentable.

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Proof sketch. We have the following pullback in $\mathbb{C}\mathbf{at}$.

$$\begin{array}{ccc}
\mathbf{Alg}(T) & \hookrightarrow & \widehat{\mathbf{Kl}(T)} \\
{\scriptstyle u_T}\downarrow & \lrcorner & \downarrow{\scriptstyle \widehat{k_T}} \\
\mathbf{Ind}(A) & \hookrightarrow & \widehat{A}
\end{array}$$

The functors $\mathbf{Ind}(A) \to \widehat{A}$ and $\widehat{\mathbf{Kl}(T)} \to \widehat{A}$ are both finitary right adjoints between locally finitely presentable categories. Thus, so are the two projection functors [Bir84].

Summary

- The pullback theorem for monads describes precisely in what sense the category of algebras is a cocompletion of the category of free algebras.
- The pullback theorem for monads generalises to a pullback theorem for relative monads with dense roots.
- This has fruitful connections to the theory of algebraic theories and cocontinuous monads.

A paper on this topic is forthcoming. In the meantime, if you found this talk interesting, you may also be interested in:

1. *Monadic and Higher-Order Structure* [Ark22]
2. *The formal theory of relative monads* [AM23a]
3. *Relative monadicity* [AM23b]

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