The theory of relative (co)monads

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Overview

Introduction

Relative monads were introduced by Altenkirch, Chapman, and Uustalu [ACU10; ACU15] (though various related concepts have been around in the literature much longer, cf. [Wal70; Die75]). There, the authors generalised several results about monads and adjunctions to relative monads. However, there were also many aspects of the theory of monads and adjunctions that were not treated.

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This talk will be an overview of some aspects of the theory of relative monads that are not yet well known, but are valuable tools to have at one's disposal.

Relative monads

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Definition 1 ([ACU10])

A *relative monad* comprises

1. a functor $j \colon A \to E$, the *root*;
2. a functor $t \colon A \to E$, the *carrier*;
3. a natural transformation $\eta \colon j \Rightarrow t$, the *unit*;
4. a form $\dagger \colon E(j, t) \Rightarrow E(t, t)$, the *extension operator*,

satisfying unitality and associativity laws.

When $j = 1$, this is (non-obviously) equivalent to the definition of monad.

Extension operators

Explicitly, an extension operator gives an assignment, taking each morphism

$$jx \xrightarrow{f} ty$$

to a morphism

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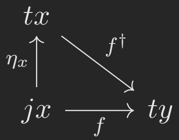
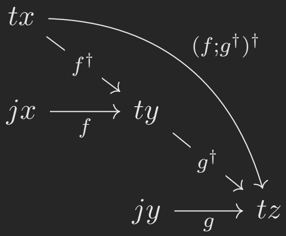
such that $(\eta_x)^\dagger = 1_{tx}$ and the following diagrams commute.





Examples of relative monads

Relative monads are abundant in category theory.

- Monads.

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- Partial monads.
- Graded monads [MU22].
- Cocontinuous monads on cocompletions (e.g. finitary monads on locally finitely presentable categories).
- Monads arising from monad–theory correspondences.

Relative monads as monoids

Distributors

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For every locally small category A, there is an identity distributor $A(1, 1) \colon A \nrightarrow A$ given by the hom-sets of A, i.e. $A(-, -) \colon A^{\mathrm{op}} \times A \to \mathbf{Set}$.

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Every functor $f \colon A \to B$ between locally small categories defines both a representable distributor $B(1,f) \colon A \nrightarrow B$ by postcomposition:

$$B(1,f)(b,a) := B(b,fa)$$

and a corepresentable distributor $B(f,1) \colon B \nrightarrow A$ by precomposition:

$$B(f,1)(a,b) := B(fa,b)$$

Forms

A form is a multiary notion of transformation between distributors,

$$
\begin{array}{ccccccccc}
A_n & \overset{p_n}{\longrightarrow} & A_{n-1} & \overset{p_{n-1}}{\longrightarrow} & \cdots & \overset{p_2}{\longrightarrow} & A_1 & \overset{p_1}{\longrightarrow} & A_0 \\
{\scriptstyle g}\downarrow & & & & \phi & & & & \downarrow{\scriptstyle f} \\
B_n & & & & \underset{q}{\longrightarrow} & & & & B_0
\end{array}
$$

and comprises a function

$$\phi_{x_0,\ldots,x_n} \colon p_1(x_0, x_1) \times \cdots \times p_n(x_{n-1}, x_n) \to q(fx_0, gx_n)$$

for each $x_0 \in |A_0|, \ldots, x_n \in |A_n|$, satisfying certain naturality laws.

When $n = 0$ and q is the identity distributor, this is exactly a natural transformation $\phi \colon f \Rightarrow g$.

Multicategories of endo-distributors

For a given locally small category A, the distributors $A \nrightarrow A$, together with forms

$$\begin{array}{ccccccccc}
A & \xrightarrow{p_n} & A & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & A & \xrightarrow{p_1} & A \\
\| & & & & \phi & & & & \| \\
A & & & & \xrightarrow[q]{\hspace{10em}} & & & & A
\end{array}$$

form a multicategory $\mathbf{Cat}[\![A, A]\!]$.

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By restricting to representable distributors – i.e. those isomorphic to a representable distributor $A(1, f)$, for some endofunctor $f \colon A \to A$ – we obtain a sub-multicategory $\mathbf{Cat}[A, A]$.

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The multicategory $\mathbf{Cat}[A, A]$ is represented by the strict monoidal category $\mathbf{Cat}(A, A)$ of endofunctors and natural transformations. Consequently, a monoid in $\mathbf{Cat}[A, A]$ is precisely a monad on A.

Skew composition of functors

Let A and E be categories. Given functors

$$f\colon A \to E \qquad\qquad g\colon A \to E$$

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However, supposing we had some fixed functor $j^*\colon E \to A$, we could form the composite

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...However, in cases of interest, we typically have a chosen functor $j\colon A \to E$, which faces in the wrong direction.

Skew composition of distributors

Instead, we shall make use of distributors. The pair $f, g\colon A \to E$ induce representable distributors

$$E(1, f)\colon A \nrightarrow E \qquad\qquad E(1, g)\colon A \nrightarrow E$$

whilst the functor $j\colon A \to E$ induces a corepresentable distributor

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We may thus form a chain of distributors

$$A \xrightarrow{E(1,f)} E \xrightarrow{E(j,1)} A \xrightarrow{E(1,g)} E$$

which acts as a notion of composite relative to j. (Note that we never actually form the composite distributor: it is enough to consider the chain.)

Skew-multicategories of distributors

For functor $j\colon A \to E$ between locally small categories, the distributors $A \nrightarrow E$, together with forms

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form a skew-multicategory[1] $\mathbf{Cat}[\![j]\!]$.

[1] In the sense of [AM23a], which generalises the skew-multicategories of Bourke [Bou17].

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Question. What is a monoid in $\mathbf{Cat}[j]$?

Intuition for $\mathbf{Cat}[j]$

Conceptually, we can think of $\mathbf{Cat}[j]$ as being forms between functors $A \to E$, where we facilitate the "composition" of two functors $A \to E$ by inserting a $j^* := E(j, 1)$ where necessary.

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It is easy to see that this recovers our multicategory $\mathbf{Cat}[A, A]$ when we specialise to $j = 1_A$.

Monoids in $\mathbf{Cat}[j]$

A monoid in $\mathbf{Cat}[j]$ comprises

1. a functor $t \colon A \to E$, the *carrier*;
2. a natural transformation $\eta \colon j \Rightarrow t$, the *unit*;
3. a form $\mu \colon E(1, t), E(j, 1), E(1, t) \Rightarrow E(1, t)$, the *multiplication*,

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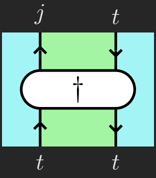
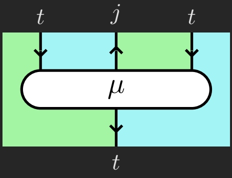
This looks remarkably similar to the definition of a relative monad.



Extension operator

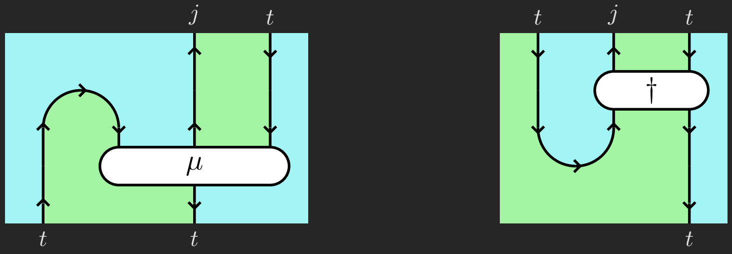


Multiplication

The calculus of (co)representable distributors

The representable and corepresentable distributors associated to a functor $f\colon A \to E$ are adjoint to one another as distributors: i.e. $E(1,f) \dashv E(f,1)$ in $\mathbf{Dist}$. From a string diagrammatic perspective, this allows us to bend arrows corresponding to representables and corepresentables (so long as they never point left).

We may define an extension operator from a multiplication and vice versa by bending strings:



Relative monads as monoids

Theorem 2

The category of monoids in the skew-multicategory $\mathbf{Cat}[j]$ *is isomorphic to the category* $\mathbf{RMnd}(j)$ *of* j*-relative monads.*

Note that we require no assumptions on the functor $j \colon A \to E$.

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We may ask when the skew-multicategory $\mathbf{Cat}[j]$ is represented by a skew-monoidal category.

Theorem 3

Suppose that (pointwise) left extensions along j exist. Then $\mathbf{Cat}[j]$ is representable.

We thus recover the characterisation of relative monads as monoids in a skew-monoidal category due to [ACU15].

Skewness and density

In fact, in many cases, it is not necessary to work with skew-multicategories. When j is dense, relative monads are monoids in an ordinary multicategory (which is a sub-(skew)-multicategory of $\mathbf{Cat}[j]$).

Recall that a functor $j\colon A \to E$ is dense if the nerve functor (a.k.a. restricted Yoneda embedding)

$$n_j := E(j-, -)\colon E \to \widehat{A}$$

is fully faithful. In practice, this assumption is usually satisfied.

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This is a first indication that density of j is a useful simplifying condition for the theory of relative monads.

Relative monads as cocontinuous monads

Lemma 4 ([ACU10])

Suppose that (pointwise) left extensions along j exist. Then:

1. $\mathbf{Cat}[j]$ *is left-normal if* j *is dense.*
2. $\mathbf{Cat}[j]$ *is right-normal if* j *is fully faithful.*
3. $\mathbf{Cat}[j]$ *is associative-normal if left extensions along* j *are* j*-absolute.*

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Theorem 5

Let Φ be a class of weights and let A be a locally small category admitting a Φ-cocompletion $A \hookrightarrow \Phi A$. The category of $(A \hookrightarrow \Phi A)$-relative monads is equivalent to the category of Φ-cocontinuous monads on ΦA.

Relative monads as monads in $\mathbf{Dist}$

Theorem 6

Suppose that $j\colon A \to E$ is dense. Then the category of j-relative monads is isomorphic to the category of monads in $\mathbf{Dist}$ whose underlying distributor is j-representable – i.e. of the form $E(j, t)$, for some functor $t\colon A \to E$.

In other words, when j is dense, a j-relative monad is equivalently specified by

1. a functor $t\colon A \to E$;
2. a natural transformation $\eta\colon j \Rightarrow t$;
3. a form $\mu\colon E(j, t), E(j, t) \Rightarrow E(j, t)$,

satisfying unitality and associativity laws.

This recovers Diers's characterisation of relative monads [Die75].

Relative adjunctions

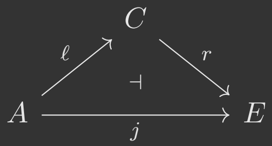
Relative adjunctions

The concept of relative adjunction is a generalisation of the concept of adjunction, where the domain of the left adjoint is permitted to be different to the codomain of the right adjoint.

Definition 7 ([Ulm68])

A *relative adjunction* comprises

1. a functor $j\colon A \to E$, the *root*;
2. a functor $\ell\colon A \to C$, the *left relative adjoint*;
3. a functor $r\colon C \to E$, the *right relative adjoint*;
4. an isomorphism of the form $C(\ell, 1) \cong E(j, r)$.



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Relative adjunctions are abundant in category theory.

- Adjunctions.
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- Nerves.
- Algebraic theories and their various generalisations [Ark22].

Properties of relative adjoints

Most of the fundamental properties of adjunctions carry across (with some modification) to relative adjunctions.

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Proposition 8 ([Ulm68])

Left relative adjoints are unique up to isomorphism. Right j-relative adjoints are unique up to isomorphism if j is dense.

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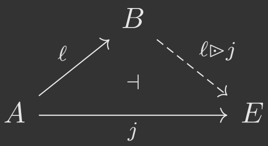
Proposition 10 ([Ulm68])

A j-relative right adjoint preserves limits when j is dense.

Constructing relative adjoints

Proposition 11

Let $j \colon A \to E$ *and* $\ell \colon A \to C$ *be functors, and suppose that* j *is fully faithful.*



1. *Suppose that the left extension* $\ell \triangleright j$ *exists and is* j*-absolute. Then* $\ell \;{}_j\!\dashv \ell \triangleright j$.
2. *Suppose that* j *is dense and that* ℓ *has a right* j*-adjoint* r*. Then* r *exhibits the left extension* $\ell \triangleright j$ *and this extension is* j*-absolute.*

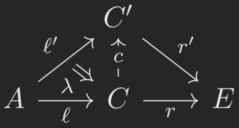
Relative adjunctions and relative monads

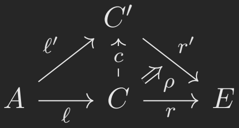
Proposition 12

Every relative adjunction induces a relative monad. Furthermore, this process extends to functors

$$\obslash_j \colon \mathbf{RAdj}_L(j)^{\mathrm{op}} \to \mathbf{RMnd}(j)$$
$$\oslash_j \colon \mathbf{RAdj}_R(j) \to \mathbf{RMnd}(j)$$

$\mathbf{RAdj}_L(j)$ (respectively $\mathbf{RAdj}_R$) is the category of j-relative adjunctions and left-morphisms (respectively right-morphisms).

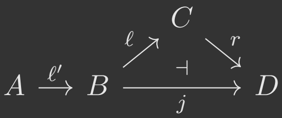




Precomposition

Proposition 13

Every relative adjunction of the form:



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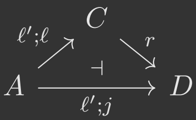
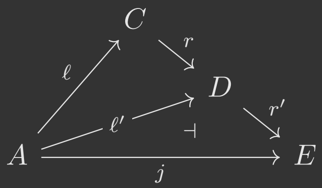


Pasting

Relative adjunctions satisfy a pasting law similar to that for pullbacks.

Proposition 14

Consider the following diagram.



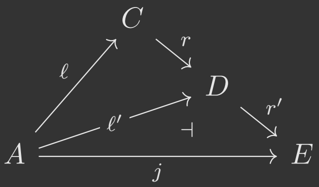
The left triangle is a relative adjunction (i.e. $\ell \;{}_{\ell'}\!\dashv r$*) if and only if the outer triangle is a relative adjunction (i.e.* $\ell \;{}_{j}\!\dashv r \,;\, r'$*).*

Pasting

Relative adjunctions satisfy a pasting law similar to that for pullbacks.

Proposition 14

Consider the following diagram.



The left triangle is a relative adjunction (i.e. $\ell \;{}_{\ell'}\!\dashv r$*) if and only if the outer triangle is a relative adjunction (i.e.* $\ell \;{}_{j}\!\dashv r \mathbin{;} r'$*).*

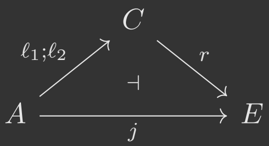
For instance, taking $j = (\ell' \mathbin{;} r')$ and r' fully faithful, this allows us to postcompose any relative adjunction by a fully faithful functor.

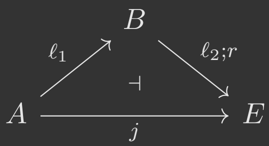
Resoluteness

A particularly useful consequence of the pasting lemma (taking $\ell' = \ell \mathbin{;} r$) gives us a way to move a functor from the left-hand side of a relative adjunction to the right-hand side.

Corollary 15

Let $(\ell_1 \mathbin{;} \ell_2) \;{}_j\!\dashv r$ be a j-adjunction. Then $\ell_1 \;{}_j\!\dashv (\ell_2 \mathbin{;} r)$ if and only if $\ell_1 \;{}_{\ell_1;\ell_2}\!\dashv \ell_2$. In this case, the two induced j-monads are isomorphic.





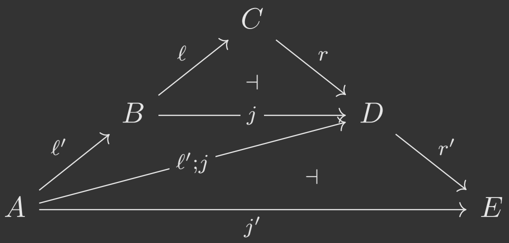
This is satisfied, for instance, if ℓ_2 is fully faithful.

Composition

Applying first the precomposition law, and then the pasting law, we obtain a general composition result for relative adjunctions.

Corollary 16

Let $\ell \,{}_j\!\dashv r$ *and* $\ell' \,;\, j \,{}_{j'}\!\dashv r'$ *be relative adjunctions as below.*



Then $\ell' \,;\, \ell \,{}_{j'}\!\dashv r \,;\, r'$.

The counit of a relative adjunction

There are several different formulations of non-relative adjunctions. In addition to the usual hom-set formulation, there is also the unit–counit formulation. At first glance, an analogous formulation for relative adjunctions does not seem possible: while every relative adjunction has a unit $\eta \colon j \Rightarrow (\ell \,; r)$, it is unclear how to express a counit, since we cannot compose $r \colon C \to E$ with $\ell \colon A \to C$.

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However, our insight for considering relative monads as monoids turns out to be helpful here too. While we may not compose r with ℓ directly, we may "compose" r with ℓ, relative to the corepresentable distributor $E(j, 1) \colon E \nrightarrow A$.

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However, our insight for considering relative monads as monoids turns out to be helpful here too. While we may not compose r with ℓ directly, we may "compose" r with ℓ, relative to the corepresentable distributor $E(j, 1) \colon E \nrightarrow A$.

A j-relative adjunction is then equivalently specified by functors $\ell \colon A \to C$ and $r \colon C \to E$, together with a unit natural transformation $\eta \colon j \Rightarrow (\ell \mathbin{;} r)$, and a counit form $\varepsilon \colon C(1, \ell), E(j, 1), E(1, r) \Rightarrow C(1, 1)$, satisfying two triangle laws.

Algebras and opalgebras

Eilenberg–Moore categories

Definition 17 ([ACU10])

Let T be a $(j\colon A \to E)$-relative monad. The *Eilenberg–Moore category* for T has as objects pairs $(e, \rtimes)$ of an object $e \in E$ and a form $\rtimes\colon E(j, e) \Rightarrow E(t, e)$ satisfying unitality and extension operator laws. Morphisms are morphisms of E commuting with the algebra structures.

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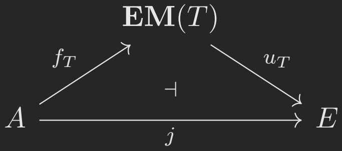
The Eilenberg–Moore category for a relative monad induces a relative adjunction,



where $f_T = a \mapsto ta$ and $u_T = (e, \rtimes) \mapsto e$.

The universal property of the Eilenberg–Moore category

Theorem 18

The construction of the Eilenberg–Moore category for a j-relative monad is right adjoint to the construction of a j-monad from a relative adjunction.

$$\mathbf{RAdj}_L(j) \underset{f_{(-)}\ {}_j\!\dashv\ u_{(-)}}{\overset{\oslash_j}{\underset{\perp}{\rightleftarrows}}} \mathbf{RMnd}(j)^{\mathrm{op}}$$

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Corollary 19 ([ACU10])

The Eilenberg–Moore resolution $f_T\ {}_j\!\dashv u_T$ is the terminal resolution of T.

Relative monad morphisms and slices

Corollary 20

The functor $u_{(-)} : \mathbf{RMnd}(j)^{\mathrm{op}} \to \mathbf{CAT}/E$ *is fully faithful.*

Relative monad morphisms and slices

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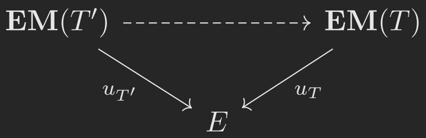
In other words, relative monad morphisms $T \to T'$ are (contravariantly) in natural bijection with functors between their categories of algebras, commuting with the forgetful functors.



Algebras

While we typically think of algebras for monads as being objects of a category, there is a more general notion of algebra structure on a functor (this notion of algebra is also called a left-module).

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While we typically think of algebras for monads as being objects of a category, there is a more general notion of algebra structure on a functor (this notion of algebra is also called a left-module). There is a corresponding generalisation for relative monads.

Definition 21

A T-algebra (a.k.a. left T-module) is a functor $e \colon D \to E$ equipped with a form $\rtimes \colon E(j, e) \Rightarrow E(t, e)$ satisfying unitality and extension operator laws.

Theorem 22

The Eilenberg–Moore category for a relative monad T is the universal T-algebra.

Kleisli categories

Definition 23 ([ACU10])

Let T be a $(j\colon A \to E)$-relative monad. The *Kleisli category* for T has the same objects as A, and $\mathbf{Kl}(T)(a, a') := E(ja, ta')$.

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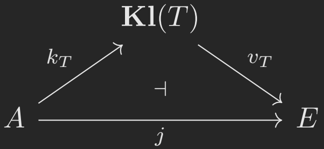
The Kleisli category for a relative monad induces a relative adjunction,



where k_T is identity-on-objects and $v_T = a \mapsto ta$.

The universal property of the Kleisli category

Theorem 24

The construction of the Kleisli category for a j-relative monad is left adjoint to the construction of a j-monad from a relative adjunction.

$$\mathbf{RAdj}_R(j) \underset{\oslash_j}{\overset{k_{(-)} \; {}_j\!\dashv \; v_{(-)}}{\underset{\perp}{\leftrightarrows}}} \mathbf{RMnd}(j)$$

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The construction of the Kleisli category for a j-relative monad is left adjoint to the construction of a j-monad from a relative adjunction.

$$\mathbf{RAdj}_R(j) \underset{\oslash_j}{\overset{k_{(-)}\ {}_j\!\dashv\ v_{(-)}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{RMnd}(j)$$

Corollary 25 ([ACU10])

The Kleisli resolution $k_T\ {}_j\!\dashv v_T$ is the initial resolution of T.

Relative monad morphisms and coslices

Corollary 26

If j is dense, then the functor $k_{(-)} \colon \mathbf{RMnd}(j) \to A/\mathbf{CAT}$ is fully faithful.

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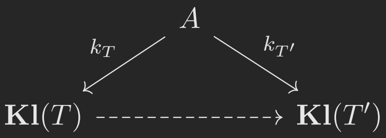
In other words, for a dense functor j, j-relative monad morphisms $T \to T'$ are in natural bijection with functors between their Kleisli categories, commuting with the inclusion functors.



Opalgebras

Just as the notion of algebra admits a generalisation from objects to functors, so too does the notion of opalgebra (i.e. an object of the Kleisli category) admit a generalisation from objects to functors.

Definition 27

A T-opalgebra (a.k.a. right T-module) is a functor $a \colon A \to B$ equipped with a form $\ltimes \colon E(j, t) \Rightarrow B(a, a)$ satisfying unitality and extension operator laws.

Theorem 28

The Kleisli category for a relative monad T is the universal T-opalgebra.

Relative (op)monadicity

Relative monadicity

Definition 29

Let $j \colon A \to E$ be a functor. A colimit in E is j-absolute if it is preserved by the nerve functor $n_j := E(j-, -) \colon E \to \widehat{A}$.

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Theorem 30

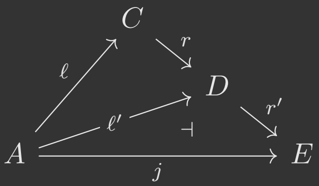
Let $j \colon A \to E$ be a dense functor. A functor $r \colon D \to E$ exhibits its domain as isomorphic to the Eilenberg–Moore category for a j-relative monad if and only if it admits a left j-relative adjoint and strictly creates j-absolute colimits.

(This is a Paré-style monadicity theorem [Par71], rather than a Beck-style monadicity theorem [Bec66].)

Monadic pasting

Theorem 31

Consider the following diagram, in which $\ell' \;{}_j\!\dashv r'$ is j-monadic.



Then the left triangle is ℓ'-monadic if and only if the outer triangle is j-monadic.

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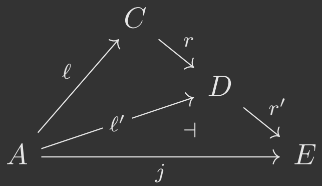

Then the left triangle is ℓ'-monadic if and only if the outer triangle is j-monadic.

In many situations of interest, this result allows us to deduce that algebraic functors (i.e. concrete functors between Eilenberg–Moore categories for relative monads) are themselves monadic.

Relative opmonadicity

While the monadicity theorem is very well known in category theory, the corresponding characterisation of opmonadic functors (i.e. those isomorphic to the Kleisli inclusion of a monad) appears less well known.

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Theorem 32

A functor $\ell \colon A \to B$ exhibits its codomain as isomorphic to the Kleisli category for a j-relative monad if and only if it admits a right j-relative adjoint and is bijective-on-objects.

Notably, in contrast to the relative monadicity theorem, this characterisation does not rely on j, apart from in the form of adjointness.

Trivial relative monads

Every functor $j\colon A \to E$ may be viewed as a trivial j-relative monad. In fact, this is the initial j-monad. While we might expect trivial relative monads to be uninteresting, just as trivial monads, this is far from being the case.

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Algebras for trivial relative monads *are* uninteresting: they are simply objects of the codomain. However, opalgebras are a different matter.

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The Kleisli category for the trivial j-monad is precisely the factorisation of j into a bijective-on-objects functor, followed by a fully faithful functor: i.e. the full image factorisation.

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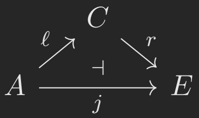
The Kleisli category for the trivial j-monad is precisely the factorisation of j into a bijective-on-objects functor, followed by a fully faithful functor: i.e. the full image factorisation. This has a useful consequence.

Proposition 33

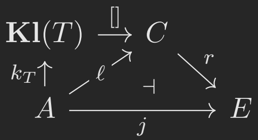
Let T be a j-monad admitting a resolution $\ell \; {}_j\!\dashv \; r$. Then $\mathbf{Kl}(T)$ is isomorphic to the Kleisli category of the trivial ℓ-monad $\mathbf{Kl}(\ell)$.

The Kleisli resolution

In other words, let T be a j-relative monad with a resolution:



Then the (bijective-on-objects, fully faithful)-factorisation of ℓ is the Kleisli category of T.



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In other words, let T be a j-relative monad with a resolution:

$$
\begin{array}{ccccc}
 & & C & & \\
 & \overset{\ell}{\nearrow} & \dashv & \searrow^{r} & \\
A & & \xrightarrow[j]{\qquad\qquad} & & E
\end{array}
$$

Then the (bijective-on-objects, fully faithful)-factorisation of ℓ is the Kleisli category of T.

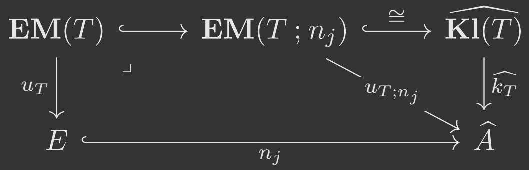
$$
\begin{array}{ccccc}
\mathbf{Kl}(T) & \xrightarrow{[]} & C & & \\
k_T \uparrow & \overset{\ell}{\nearrow} & \dashv & \searrow^{r} & \\
A & & \xrightarrow[j]{\qquad\qquad} & & E
\end{array}
$$

This gives a particularly convenient method to check whether two relative adjunctions induce the same relative monad.

The pullback theorem

Theorem 34

Let $j\colon A \to E$ be a dense functor. There is a pullback in $\mathbf{Cat}$ *as follows.*



The pullback theorem

Theorem 34

Let $j\colon A \to E$ be a dense functor. There is a pullback in $\mathbf{Cat}$ as follows.

$$
\begin{array}{ccccc}
\mathbf{EM}(T) & \hookrightarrow & \mathbf{EM}(T\,;n_j) & \xrightarrow[\cong]{} & \widehat{\mathbf{Kl}(T)} \\
\downarrow{\scriptstyle u_T} & & & & \downarrow{\scriptstyle \widehat{k_T}} \\
E & & \xhookrightarrow{\quad n_j \quad} & & \widehat{A}
\end{array}
$$

Corollary 35

The comparison functor $i_T\colon \mathbf{Kl}(T) \to \mathbf{EM}(T)$ is dense.

Duality

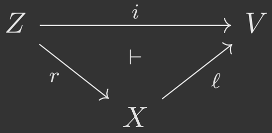
Relative coadjunctions

Unlike the notion of adjunction, the notion of relative adjunction is not self-dual.

Definition 36 ([Ulm68])

A *relative coadjunction* comprises

1. a functor $i \colon Z \to V$, the *coroot*;
2. a functor $\ell \colon X \to V$, the *left relative coadjoint*;
3. a functor $r \colon Z \to X$, the *right relative coadjoint*;
4. an isomorphism of the form $V(\ell, i) \cong X(1, r)$.



Relative comonads

Just as with monads, comonads also have a generalisation to arbitrary functors.

Definition 37

A *relative comonad* comprises

1. a functor $i\colon Z \to V$, the *coroot*;
2. a functor $d\colon Z \to V$, the *underlying functor*;
3. a natural transformation $\varepsilon\colon d \Rightarrow i$, the *counit*;
4. a form $\dagger\colon V(d, i) \Rightarrow V(d, d)$, the *coextension operator*,

satisfying counitality and coassociativity laws.

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Proposition 38

Every relative coadjunction induces a relative comonad.

The monad–theory correspondence

Algebraic theories and monads

It is well known that the category of finitary algebraic theories is equivalent to the category of finitary monads on $\mathbf{Set}$ (cf. [Lin66; Die75; Pow99]). Traditionally, the proof is somewhat involved, and usually proceeds via a monadicity theorem.

Algebraic theories and monads

It is well known that the category of finitary algebraic theories is equivalent to the category of finitary monads on $\mathbf{Set}$ (cf. [Lin66; Die75; Pow99]). Traditionally, the proof is somewhat involved, and usually proceeds via a monadicity theorem.

However, the monad–theory correspondence is actually a direct consequence of some of the properties of relative adjunctions and relative monads that we have discussed. That is, it fits within the general theory of relative monads.

This demonstrates that these abstract results about relative monads can have applications to interesting and long-standing questions in category theory.

Algebraic theories are relative adjoints

Definition 39

Denote by $\mathbb{F}$ the free category with strict finite coproducts on a single object. A *finitary algebraic theory* is an identity-on-objects functor from $\mathbb{F}$ that preserves finite coproducts.

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Lemma 40

A functor from $\mathbb{F}$ preserves finite coproducts if and only if it is left adjoint relative to $\mathbb{F} \hookrightarrow \mathbf{Set}$.

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Lemma 40

A functor from $\mathbb{F}$ preserves finite coproducts if and only if it is left adjoint relative to $\mathbb{F} \hookrightarrow \mathbf{Set}$.

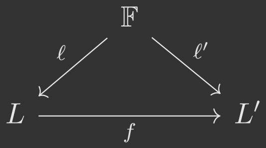
Corollary 41

A functor from $\mathbb{F}$ is a finitary algebraic theory if and only if it is the Kleisli inclusion for a $(\mathbb{F} \hookrightarrow \mathbf{Set})$-relative monad.

Algebraic theories and relative monads

Definition 42

A *morphism* of finitary algebraic theories is a morphism of coslices under $\mathbb{F}$.



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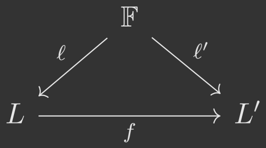

Theorem 43

The category of finitary algebraic theories is equivalent to the category of $(\mathbb{F} \hookrightarrow \mathbf{Set})$*-relative monads.*

Algebraic theories and monads

Theorem 44

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Algebraic theories and monads

Theorem 44

The category of finitary algebraic theories is equivalent to the category of finitary monads on $\mathbf{Set}$.

Corollary 45

Let $\ell\colon \mathbb{F} \to L$ *be a finitary algebraic theory. The category of algebras for the induced relative monad is given by the following pullback in* $\mathbf{Cat}$.

$$\begin{array}{ccc} \mathbf{Cart}(L^{\mathrm{op}}, \mathbf{Set}) & \hookrightarrow & \widehat{L} \\ {\scriptstyle \mathbf{Cart}(\ell^{\mathrm{op}}, \mathbf{Set})}\downarrow & \lrcorner & \downarrow{\scriptstyle \widehat{\ell}} \\ \mathbf{Cart}(\mathbb{F}^{\mathrm{op}}, \mathbf{Set}) & \hookrightarrow & \widehat{\mathbb{F}} \end{array}$$

The formal theory

Enriched relative monads

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The 2-dimensional structure that axiomatises the behaviour of (enriched) categories, functors, distributors, and forms is called a virtual equipment [CS10]. This is the setting in which we work in [AM23a; AM23b]. This permits us to capture ordinary relative monads, enriched relative monads, internal relative monads, strong relative monads, and so on, in a single framework.

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However, in our papers, we spell out examples of interest in $\mathbb{V}\text{-}\mathbf{Cat}$, so it should be approachable even for those readers not interested in the formal aspects.

Summary

This has been a very quick overview of some of the fundamental aspects of the theory of relative (co)monads. The hope is that this gives you an idea of the tools at your disposal for working with relative (co)monads and relative (co)adjunctions, and also gives a taste for how powerful these techniques are for proving theorems of practical interest.

The first two papers in our series are:

1. *The formal theory of relative monads*
2. *Relative monadicity*

Some of the results I have mentioned may also be found in my thesis:

3. *Monadic and Higher-Order Structure*

Keep an eye out for the next installments...

References I

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