Virtual double categories

Past, present, and future

Nathanael Arkor, TalTech

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This talk

My aim is to convince you that virtual double categories are fundamental in category theory, and to give you some sense for where they may appear in your own work.

To this end, I will:

- Recall the concept of a virtual double cat.
- Introduce a range of motivating examples
- Discuss some applications of their theory

Part I. Multicategories

Multicategories as generalised categories

The concept of a multicategory is typically presented as a generalisation of a category in which morphisms may have multiary input.

Every multicategory has an underlying category of unary multimorphisms.

Multicategories as generalised monoidal categories

Multicategories may also be considered generalisations of monoidal categories in which morphisms $x_1 \otimes \cdots \otimes x_n \longrightarrow y$ are represented by multimorphisms $x_1, \ldots, x_n \longrightarrow y$.

From this perspective, it is natural to view a multicategory as structure on an underlying category, rather than a stand-alone structure.

Multicategories versus monoidal categories

Every monoidal category induces a multicategory. Conversely, the multicategories that correspond to monoidal categories may be identified by a universal property.

Definition [Hermida '00]

A multicategory $\mathcal{M}$ is representable when, for every $X_1, \ldots, X_n \in \mathcal{M}$, there is $X_1 \otimes \cdots \otimes X_n \in \mathcal{M}$ such that

$$\mathcal{M}(\vec{W}, X_1, \ldots, X_n, \vec{Y}; Z) \cong \mathcal{M}(\vec{W}, X_1 \otimes \cdots \otimes X_n, \vec{Y}; Z)$$

Free monoidal categories

There is a 2-adjunction

$$\begin{tikzcd} \mathbf{Multicat} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & \mathbf{StrMonCat}_s \arrow[l, shift left=1.5, "U"] \end{tikzcd}$$

which is lax-idempotent, monadic, and comonadic [Hermida '00; Elmendorf & Mandell '09].

Consequently, the full sub-2-category of **Multicat** spanned by the representable multicategories is biequivalent to $\mathbf{MonCat}_l$.

Advantages of multicategories

In many respects, multicategories are more well-behaved than monoidal categories.

- It is often simpler to construct a multicategory and prove it is representable than to construct a monoidal category directly.
- Multicategories are closed under taking subcategories.
- $\mathbf{Multicat}$ is complete and cocomplete, unlike $\mathbf{MonCat}_l$.
- $\mathbf{Multicat}$ admits many exponentials.

Part II. Categorification

Categorification

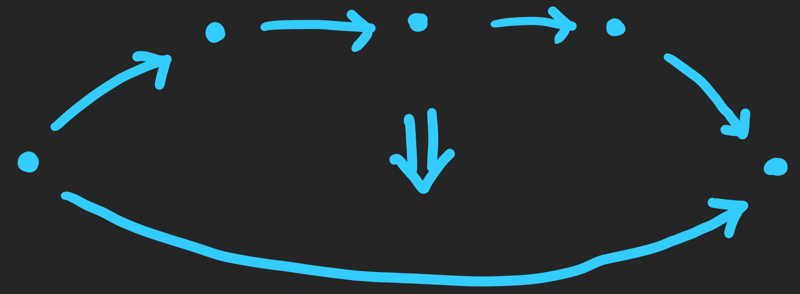
The two perspectives on multicategories — i.e. as stand-alone structures, or structure on a category — lend themselves to different generalisations.

For instance, there are many situations in which we would like to view a multicategory as some 2-dimensional structure with one object, in the same way that we may view monoidal categories as one-object bicategories.

Multibicategories

Viewing a multicategory as a stand-alone structure, the natural categorification is the notion of multibicategory (a.k.a. virtual bicategory), which has

- objects
- 1-cells
- multiary 2-cells



Virtual double categories

Viewing a multicategory as structure on a category, multibicategories are unsuitable, since they lack an underlying category.

From this perspective, the natural categorification is the notion of virtual double category.

Virtual double categories

A virtual double category [Burroni '71] comprises

- a collection of objects
- for each pair of objects, a collection of tight morphisms $X \longrightarrow Y$
- for each pair of objects, a collection of loose morphisms $X \leftarrow\!\!\!\!+\!\!\!- X'$
- for each frame, a collection of cells

$$\begin{tikzcd}
X_0 \arrow[d, "f"'] & X_1 \arrow[l, "p_1"', "\shortmid"{marking}] & \cdots \arrow[l, "p_2"', "\shortmid"{marking}] & X_n \arrow[l, "p_n"', "\shortmid"{marking}] \arrow[d, "f'"] \\
Y \arrow[rrr, phantom, "\varphi" {yshift=12pt}] & & & Y' \arrow[lll, "q", "\shortmid"{marking}]
\end{tikzcd}$$

Virtual double categories

- identity and composite tight morphisms, forming a category

- identity cells
$$\begin{tikzcd} \bullet \arrow[d, equal] \arrow[r, phantom, ""] & \bullet \arrow[l, "p"', "\shortmid" marking] \arrow[d, equal] \arrow[dl, phantom, "="] \\ \bullet & \bullet \arrow[l, "p", "\shortmid" marking] \end{tikzcd}$$

- composite cells

$$\begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l, "\shortmid" marking] \arrow[d, phantom, "\varphi_1"] & \bullet \arrow[l, "\shortmid" marking] \arrow[d] & \bullet \arrow[l, "\shortmid" marking] \arrow[d, phantom, "\varphi_2"] & \bullet \arrow[l, "\shortmid" marking] \arrow[d] \\ \bullet \arrow[d] & & \bullet \arrow[ll, "\shortmid" marking] \arrow[d, phantom, "\psi"] & & \bullet \arrow[ll, "\shortmid" marking] \arrow[d] \\ \bullet & & & & \bullet \arrow[llll, "\shortmid" marking] \end{tikzcd} \longmapsto \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l, "\shortmid" marking] & \bullet \arrow[l, "\shortmid" marking] \arrow[d, phantom, "(\varphi_1, \varphi_2) ; \psi"] & \bullet \arrow[l, "\shortmid" marking] & \bullet \arrow[l, "\shortmid" marking] \arrow[d] \\ \bullet & & & & \bullet \arrow[llll, "\shortmid" marking] \end{tikzcd}$$

Part III. Examples

Special cases

1. A VDC with no loose morphisms (hence no cells) is precisely a category.
2. A VDC with one object and one tight morphism is precisely a multicategory.
3. A VDC with no non-identity tight morphisms is precisely a multibicategory.

Cospans

[Burroni '71]

For every category $\mathcal{E}$, there is a VDC $\mathbb{C}\mathrm{ospan}(\mathcal{E})$ whose category of objects and tight morphisms is $\mathcal{E}$, for which a loose morphism $X \nrightarrow Y$ is a cospan

$$X \to A \leftarrow Y$$

and for which a cell (left) is a family of morphisms in $\mathcal{E}$ (right) making the diagram commute.

$$\begin{tikzcd}
X_0 \arrow[d, "f"'] & X_1 \arrow[l, "p_1"', "\shortmid" marking] & \cdots \arrow[l, "p_2"', "\shortmid" marking] & X_n \arrow[l, "p_n"', "\shortmid" marking] \arrow[d, "f'"] \\
Y & & & Y' \arrow[lll, "q", "\shortmid" marking]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f"'] & P_1 \arrow[drr] & X_1 \arrow[l] \arrow[r] & P_2 \arrow[d] & \cdots \arrow[l] \arrow[dl, dotted] & X_n \arrow[l] \arrow[d, "f'"] \\
Y \arrow[rrr] & & & B & & Y' \arrow[ll]
\end{tikzcd}$$

Spans

For every category $\mathcal{E}$ with pullbacks, there is a VDC $\mathbb{S}\mathrm{pan}(\mathcal{E})$ whose category of objects and tight morphisms is $\mathcal{E}$, for which a loose morphism $X \nrightarrow Y$ is a span

$$X \leftarrow A \rightarrow Y$$

and for which a cell (left) is a span morphism (right) from the iterated pullback.

$$
\begin{tikzcd}
X_0 \arrow[d, "f"'] & X_1 \arrow[l, "p_1"', "\shortmid"{marking}] & \cdots \arrow[l, "p_2"', "\shortmid"{marking}] & X_n \arrow[l, "p_n"', "\shortmid"{marking}] \arrow[d, "f'"] \\
Y & & & Y' \arrow[lll, "q", "\shortmid"{marking}]
\end{tikzcd}
\qquad
\begin{tikzcd}
& & & P \arrow[dll, bend right] \arrow[d] \arrow[ddd, dashed, bend left] \arrow[drr, bend left] & & \\
X_0 & P_1 \arrow[l] \arrow[r] & X_1 & P_2 \arrow[l] \arrow[r] & \cdots \arrow[r] & X_n \\
Y \arrow[u, "f"] & & & & & Y' \arrow[u, "f'"'] \\
& & & B \arrow[ulll] \arrow[urr] & &
\end{tikzcd}
$$

Categories and distributors

[Burroni '71]

There is a VDC $\mathbb{D}\mathrm{ist}$ for which

- an object is a (locally small) category
- a tight morphism is a functor
- a loose morphism $X \nrightarrow Y$ is a distributor, i.e. a functor $Y^{op} \times X \to \mathrm{Set}$
- a cell

$$
\begin{tikzcd}
X_0 \arrow[d, "f"'] & X_1 \arrow[l, "p_1"'] & \cdots \arrow[l, "p_2"'] & X_n \arrow[l, "p_n"'] \arrow[d, "f'"] \\
Y & & & Y' \arrow[lll, "q"]
\end{tikzcd}
$$

is a natural family of functions

$$\left\{ p_1(x_0, x_1), \ldots, p_n(x_{n-1}, x_n) \longrightarrow q(f(x_0), f'(x_n)) \right\}_{\vec{x}}$$

Matrices

[Leinster '04]

For a multicategory $\mathcal{V}$, there is a VDC $\mathcal{V}$-$\mathbb{M}\mathrm{at}$ in which an object is a class, a tight morphism is a function, a loose morphism $X \xrightarrow{\shortmid} Y$ is a family $\{ p(y,x) \in \mathcal{V} \}_{x \in X, y \in Y}$ and for which a cell

$$
\begin{tikzcd}
X_0 \arrow[d, "f"'] & X_1 \arrow[l, "p_1"'] & \cdots \arrow[l, "p_2"'] & X_n \arrow[l, "p_n"'] \arrow[d, "f'"] \\
Y & & & Y' \arrow[lll, "q"]
\end{tikzcd}
$$

is a family of multimorphisms in $\mathcal{V}$

$$\{ p_1(x_0, x_1), \ldots, p_n(x_{n-1}, x_n) \longrightarrow q(f(x_0), f'(x_n)) \}_{\vec{x}}$$

Internal and enriched categories

For every category $\mathcal{E}$ with pullbacks, there is a VDC $\mathbb{D}\mathrm{ist}(\mathcal{E})$ of categories internal to $\mathcal{E}$, their functors, distributors, and natural transformations.

For every monoidal category $\mathcal{V}$, there is a VDC $\mathcal{V}\text{-}\mathbb{D}\mathrm{ist}$ of categories enriched in $\mathcal{V}$, their functors, distributors, and natural transformations.

Double categories

There is a VDC $\mathbb{D}\mathrm{bl}$ for which

- an object is a weak double category
- a tight morphism is a strict functor
- a loose morphism $\mathbb{X} \looparrowright \mathbb{Y}$ is a tight distributor, i.e. a lax functor

$$\mathbb{Y}^{\mathrm{opt}} \times \mathbb{X} \longrightarrow \mathbb{S}\mathrm{pan}$$

- a cell is a transformation.

There is a similar VDC $\mathbb{V}\mathbb{D}\mathbb{C}$ of VDCs.

Part IV. Representability

Loose identities and loose composites

A VDC does not admit identity or composite loose morphisms in general. However, their existence may be characterised by a universal property.

- A loose identity on an object X comprises a loose morphism $X \xleftarrow{X(1,1)} X$ and a nullary cell

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\iota_X"] & X \arrow[d, equal] \\
X & X \arrow[l, "|" marking, "X(1,1)"]
\end{tikzcd}
$$

such that 'mapping out of $X(1,1)$ is the same as mapping out of nothing'.

Loose identities and loose composites

- A loose composite of a chain (left)

$$X_0 \xleftarrow{p_1} \cdots \xleftarrow{p_n} X_n \qquad\qquad X_0 \xleftarrow{p_1 \odot \cdots \odot p_n} X_n$$

comprises a loose morphism (right) and a cell

$$\begin{tikzcd} X_0 & \cdots \arrow[l, "p_1"'] & X_n \arrow[l, "p_n"'] \\ X_0 \arrow[u, equal] & & X_n \arrow[ll, "p_1 \odot \cdots \odot p_n"] \arrow[u, equal] \arrow[ll, phantom, shift left=4ex, "\gamma_{p_1,\ldots,p_n}"] \end{tikzcd}$$

such that 'mapping out of $p_1 \odot \cdots \odot p_n$ is the same as mapping out of $p_1, \ldots, p_n$'.

Normality and representability

A VDC is called normal when it admits all loose identities, and representable when it admits all loose composites.

- A VDC with loose identities and no other loose morphisms is precisely a 2-category.
- More generally, every normal VDC has an underlying 2-category.
- A representable VDC is precisely a weak double category.

Composites in examples

- $\mathbb{C}\mathrm{ospan}(\mathcal{E})$ is always normal, but admits binary composites only when $\mathcal{E}$ has pushouts.
- $\mathbb{S}\mathrm{pan}(\mathcal{E})$ is always representable.
- $\mathcal{V}$-$\mathbb{M}\mathrm{at}$ admits composites if $\mathcal{V}$ admits coproducts preserved by the tensor product.
- $\mathbb{D}\mathrm{ist}$ is always normal, but only admits the composite $\mathcal{C} \mathrel{\mapstochar\rightarrow} \mathcal{D} \mathrel{\mapstochar\rightarrow} \mathcal{E}$ if $\mathcal{D}$ is small.

Composites in examples

- $\mathbb{D}\mathrm{ist}(\mathcal{E})$ is always normal, but admits binary composites only when $\mathcal{E}$ admits reflexive coequalisers preserved by binary products.
- $\mathcal{V}\text{-}\mathbb{D}\mathrm{ist}$ is always normal, but only admits the composite $\mathcal{C} \nrightarrow \mathcal{D} \nrightarrow \mathcal{E}$ if $\mathcal{V}$ admits a certain coend.
- $\mathbb{D}\mathrm{bl}$ is normal, but not representable.

Representability

In many situations, we are interested in weak double categories, where composites of loose morphisms do exist, but establishing coherence involves tedious or intricate calculations.

In these situations, it is often simpler to first construct a VDC, and then prove it is representable.

This is particulary true when composition is defined via a universal property (i.e. usually).

Lax functors

[DPP '06]

VDCs and weak double categories are related by a lax-idempotent pseudoadjunction:

$$\begin{tikzcd} \mathbf{VDbl} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{WDbl}_s \arrow[l, shift left, "U"] \end{tikzcd}$$

Consequently, the sub-2-category of $\mathbf{VDbl}$ spanned by the representable VDCs is biequivalent to $\mathbf{WDbl}_l$, the 2-category of weak double categories and lax functors.

Unlike $\mathbf{WDbl}_l$, $\mathbf{VDbl}$ is very well-behaved: for instance, it is complete and cocomplete.

Part V. Applications

Part V. Applications

Monads and bimodules

Monads and bimodules

One of the most important constructions in two-dimensional category theory is the Mod construction, which takes a bicategory $\mathcal{K}$ and produces a new bicategory, whose objects are monads in $\mathcal{K}$ and whose 1-cells $S \longrightarrow T$ are S-T bimodules.

Example. $\mathrm{Mod}(\mathrm{Span})$ is the bicategory of small categories and distributors.

Shortcomings with Mod

Unfortunately, there are several drawbacks.

- We cannot capture functors between categories.
- k is required to have local reflexive coequalisers.
- Mod does not admit a convenient universal property.

The Mod construction

[B. '71, L. '04]

For any VDC $\mathbb{X}$, there is a VDC $\mathbf{Mod}(\mathbb{X})$ whose

- objects are monads in $\mathbb{X}$
- tight morphisms are monad morphisms
- loose morphisms are monad bimodules
- cells are monad transformations

Examples

- $\mathbf{Mod}(\mathbb{S}\mathrm{pan}(\mathcal{E})) \cong \mathbb{D}\mathrm{ist}(\mathcal{E})$
- $\mathbf{Mod}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) \cong \mathcal{V}\text{-}\mathbb{D}\mathrm{ist}$

(More to follow.)

Cofree normal VDCs

The $\mathbb{M}\mathrm{od}$ construction admits a very useful universal property.

Theorem [Cruttwell & Shulman '10]
There is a pseudoadjunction:
$$\begin{tikzcd} \mathsf{VDbl}_n \arrow[r, shift left=2, "U"] \arrow[r, phantom, "\perp"] & \mathsf{VDbl} \arrow[l, shift left=2, "\mathbb{M}\mathrm{od}"] \end{tikzcd}$$

Corollary [Bénabou]
Normal lax functors into $\mathbb{D}\mathrm{ist}$ correspond to arbitrary lax functors into $\mathbb{S}\mathrm{pan}$.

Part V. Applications

Enrichment

Enrichment of categories

In what structures may we enrich a category?

- The classical answer is a monoidal category [Bénabou '65; Maranda '65].
- More generally, we may enrich in a multicategory [Lambek '69].
- Walters observed that sheaves may be viewed as categories enriched in a bicategory ['81].
- Leinster gave examples of enrichment in virtual double categories, and argues this is maximally general ['02].

Enrichment in a VDC

An advantage of enrichment in a VDC is that the base of enrichment $\mathbb{V}$ forms the same structure as $\mathbb{V}\text{-}\mathbb{D}\mathrm{ist}$. This allows us to iterate the construction of enriched categories.

$$(-)\text{-}\mathbb{D}\mathrm{ist} : \mathrm{VDbl} \longrightarrow \mathrm{VDbl}$$

It is natural to wonder if this forms part of a (two-dimensional) monad on VDbl and, if so, what its algebras are.

Enriched categories as a free cocompletion

Theorem [Arkor]

$(-)\text{-}\mathbb{D}\mathrm{ist}$ forms a lax-idempotent 2-monad on $\mathbf{VDbl}_n$, whose algebras are the normal VDCs admitting collages of enriched categories (a kind of two-dimensional colimit).

This generalises an earlier theorem of Garner & Shulman ['16], which imposes constraints on $\mathbb{V}$.

Part V. Applications

Generalised multicategories

Generalised multicategories

A. Burroni introduced a framework for studying generalised forms of internal multicategories, in which the domains of multimorphisms are not necessarily lists of objects, but are instead parameterised by a monad ['71].

Given a monad T on a category $\mathcal{E}$ with pullbacks, a T-category is a span equipped with 'monad structure'.

$$T\mathcal{C}_0 \xleftarrow{s} \mathcal{C}_1 \xrightarrow{t} \mathcal{C}_0$$

VDCs as generalised multicategories

Category	Monad	T-category
Set	Identity	Category
Set	Monoid	Multicategory
Grph	Category	VDC

This gives further evidence that VDCs are the appropriate categorification of multicategories.

A framework for generalised multicategories

Cruttwell & Shulman ['10] extended Burroni's setting (as well as later work [Leinster '04; Hermida '00]) to the setting of a monad on a virtual double category.

$$\mathbb{X} \circlearrowleft T \mapsto \mathbb{L}\mathrm{Kl}(T) \longmapsto n\mathbb{M}\mathrm{od}(\mathbb{L}\mathrm{Kl}(T))$$

This permits them to obtain not just generalised multicategories and functors, but also bimodules.

Part V. Applications

Universal properties

A good setting for 2D universality

We have already seen two examples where relaxing assumptions, and working with VDCs rather than bicategories or double categories, made it possible to obtain much cleaner universal properties.

- **Mod** (cofree normal VDC)
- **Dist** (free collage cocompletion)

These are far from isolated instances.

Restrictions

In many VDCs, we can restrict a loose morphism along a pair of tight morphisms:

$$A \xrightarrow{f} B \overset{p}{\leftarrow\!\!\!+\!\!\!-} C \xleftarrow{g} D \quad \mapsto \quad A \overset{p(f,g)}{\leftarrow\!\!\!-\!\!\!+\!\!\!-} D$$

For instance, in $\mathbb{D}\mathrm{ist}$, such a restriction is given by $(a,d) \mapsto p(f(a), g(d))$.

Restriction satisfies an appropriate universal property.

Virtual equipments

A virtual equipment is a VDC with

- loose identities (i.e. a normal VDC)
- restrictions.

Examples: $\mathbb{C}\mathrm{ospan}(\mathcal{E})$, $\mathbb{S}\mathrm{pan}(\mathcal{E})$, $\mathbb{D}\mathrm{ist}(\mathcal{E})$, $\mathcal{V}\text{-}\mathbb{D}\mathrm{ist}$.

In a virtual equipment, every tight morphism $A \xrightarrow{f} B$ admits a companion $A \xrightarrow{f_*} B$ and a conjoint $B \xrightarrow{f^*} A$.

The universal property of cospans

Theorem [Dawson, Paré & Pronk '10]

For each category $\mathcal{E}$, the VDC $\mathbb{C}\mathrm{ospan}(\mathcal{E})$ is the free virtual equipment on $\mathcal{E}$.

Dually, $\mathbb{S}\mathrm{pan}(\mathcal{E})$ is the free 'co-virtual co-equipment' on $\mathcal{E}$.

$\mathbb{P}\mathrm{oly}(\mathcal{E})$ admits an analogous co-virtual universal property [Arkor & Clarke].

Part V. Applications

Yoneda theory

Yoneda theory for double categories

The Yoneda lemma is of great importance in both one- and two-dimensional category theory.

It is thus desirable to have such a result for double categories. Paré investigated this question, determining that the appropriate notion of presheaf on a weak double category $\mathbb{D}$ is a lax functor from $\mathbb{D}^{opt}$ to $\mathbb{S}\mathrm{pan}$ ['10].

However, lax functors do not assemble into a weak double category, but merely a virtual double category.

Exponentiability

Unlike the category of categories, the category of multicategories is not cartesian closed.

However, the exponentiable multicategories admit an elegant characterisation: they are precisely the promonoidal categories [Pisani '14].

This gives a conceptual explanation for the convolution monoidal structure on functor categories described by Day ['70].

Exponentiable VDCs

We may carry out a similar analysis for VDCs, in their capacity as 'many-object multicategories'.

Theorem [Arkor]

Every representable VDC (i.e. weak double category) is exponentiable. Furthermore, $\mathbf{Mod}(\mathbb{X}^{\mathbb{A}})$ is equivalent to Paré's VDC $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{X})$.

This exhibits a universal property that significantly simplifies the Yoneda theory.

Convolution for double categories

Behr, Melliès & Zeilberger showed that, for each small weak double category $\mathbb{D}$, the functor category $[\mathbb{D}_1, \mathbf{Set}]$ obtains a colax monoidal structure ['23].

In fact, this is precisely $(\Sigma\mathbf{Set})^{\mathbb{D}}$, where $\Sigma\mathbf{Set}$ is the one-object VDC given by delooping the monoidal category $\mathbf{Set}$.

Part V. Applications

Formal category theory

A proliferation of category theories

There are many flavours of category theory.

- Ordinary
- Enriched
- Internal
- Monoidal
- Fibred
- Indexed
- Multi-
- Double
- Parameterised

...

In each, we find the same kinds of definitions and theorems.

- Presheaves & cocompletions
- Adjointness
- Monadicity
- Presentability

...

Formal category theory

Formal category theory (FCT) applies the philosophy of category theory to category theory itself, by identifying shared structure in each of these examples, to provide a common framework in which to establish category theoretic theorems just once, thereby obtaining the specific variants by specialising to examples.

Frameworks for FCT

Several approaches to FCT have been proposed in the literature.

- 2-categories with properties (e.g. limits, closure) [Gray '74]
- Yoneda structures (motivated by the presheaf construction) [Street & Walters '78]
- Proarrow equipments (motivated by distributors) [Wood '82]

Frameworks for FCT

- Lax-idempotent pseudomonads (motivated by free cocompletions) [Bunge & Funk '99]
- Virtual equipments (motivated by distributors) [Cruttwell & Shulman '10]
- Augmented virtual equipments (motivated by distributors) [Koudenburg '24]

Virtual equipments

Recall that a virtual equipment (VE) is a VDC admitting loose identities and restrictions.

This setting has a number of advantages for FCT over previous settings.

- It does not require comma objects or cartesian closure (e.g. $\mathcal{V}$-$\mathbb{C}\mathrm{at}$).
- It does not require a presheaf construction (e.g. $\mathbb{C}\mathrm{at}(\mathcal{E})$ or $\mathcal{V}$-$\mathbb{C}\mathrm{at}$ for $\mathcal{V}$ not closed).
- It does not require free cocompletions.

Virtual equipments

- It does not require distributors to be composable (e.g. large categories, $\mathcal{V}$-$\mathbb{C}$at for $\mathcal{V}$ not cocomplete, double categories).

Augmented VEs also share these advantages, but increase complexity in some aspects, while reducing complexity in others.

FCT with virtual equipments

In the last decade, a significant amount of FCT has been developed in the setting of VEs.

- Limits and colimits
- (Relative) monads and (relative) adjunctions
  - Monadicity
  - Nerve theorem
- Presheaf constructions
- Totality

The formal theory of relative monads

Two fundamental concepts in category theory are monads and adjunctions. Their theory has been well developed in ordinary category theory, but much less so in other settings, including in enriched category theory.

Street showed that aspects of the theory of monads could be carried out in a 2-category ['72]. However, many aspects require a richer setting.

A formal monadicity theorem

Theorem [Arkor & McDermott '25]

Let $D \xrightarrow{u} E$ be a tight morphism in a VE admitting algebra objects for monads.

u is monadic iff u admits a left adjoint and creates absolute colimits.

In particular, this holds for enrichment in an arbitrary monoidal category.

The formal theories of presheaves and cocompletions

Two of the most fundamental constructions in category theory are the category of presheaves $[\mathscr{C}^{op}, \mathbf{Set}]$ on a small category $\mathscr{C}$, and the free cocompletion of a category under a class Φ of colimits.

Theorem [Arkor & McDermott]

Under minimal assumptions on a class Φ of weights in a VE, if a Φ-presheaf object exists, it exhibits a free Φ-cocompletion.

Part VI. Future directions

New perspectives on two-dimensional category theory

In the last 50 years, the theory of 2-categories and bicategories has been developed substantially. As we have seen, reinterpreting these results from the perspective of virtual double categories often leads to new insights, in addition to placing everything in a common framework.

However, there are many topics left to treat.

Some important directions

Below are some topics that I hope to see developed (further) in the coming years.

- Limits & colimits in VDCs (cf. [Kawase '25])
- Monads on VDCs (cf. [Koudenburg])
- Formal category theory in symmetric VDCs (e.g. coend calculus)
- Generalised polycategories (cf. [Burroni '73])
- Enrichment in VDCs
- Three-dimensional structure

Part VII. Conclusion

Summary

- Virtual double categories are an expressive class of two-dimensional structures that subsume double categories and multicategories.
- The theory of VDCs is typically richer and more well-behaved than that of 2-/bicategories.
- VDCs provide an ideal setting in which to study formal category theory.
- Much remains to be done...

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